

CHARACTERISTIC CYCLES AND THE MICROLOCAL GEOMETRY OF THE GAUSS MAP

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ABSTRACT. We study Tannaka groups attached to holonomic \mathcal{D} -modules on abelian varieties. The Fourier-Mukai transform leads to an interpretation in terms of principal bundles which implies that the arising groups are almost connected, while microlocal constructions relate multiplicative subgroups to Gauss maps of characteristic cycles. This provides a link between monodromy and Weyl groups that gives a uniform approach to all known examples, and it explains the occurrence of minuscule representations. We illustrate our results with a new Torelli theorem for subvarieties and with a Tannakian obstruction for a subvariety to be a sum of small-dimensional varieties. In an appendix we sketch similar constructions for twistor modules which may be useful for examples that are not of geometric origin.

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A common thread in algebraic geometry, algebraic analysis and arithmetic is the study of algebraic varieties by various realizations of the six functor formalism such as holonomic \mathcal{D} -modules, Hodge modules, twistor modules or ℓ -adic perverse sheaves. In the case of abelian varieties, a rich supply of examples is obtained from the convolution product that leads to a Tannakian correspondence between objects of the above type and representations of certain algebraic groups [26] [21]. From an arithmetic point of view this is an analog of the approach to the Mellin transform in [13] [20], and it is related to generic vanishing theorems over finite fields [47], but there also is a more geometric aspect: In particular, to any closed subvariety of an abelian variety one may attach a reductive algebraic group in a natural way by applying the Tannakian correspondence to the intersection cohomology module on the subvariety. While these groups are closely connected with the moduli of abelian varieties and their subvarieties [25] [23] [22], only very little is known about

2010 *Mathematics Subject Classification.* Primary 14K12; Secondary 14F10, 18D10.

Key words and phrases. Abelian variety, tensor category, convolution product, \mathcal{D} -module, Gauss map, Fourier-Mukai transform, microlocalization, twistor module.

their structure in general. The example of intermediate Jacobians of smooth cubic threefolds gives a hint: It leads to an exceptional group of type E_6 whose Weyl group is the Galois group of the 27 lines on a smooth cubic surface. Motivated by this observation, it has been conjectured in [23] that there should be a general relation between the Weyl groups arising from the Tannakian correspondence and the monodromy of certain Gauss maps. One goal of this paper is to establish such a relation in the framework of holonomic \mathcal{D} -modules (corollary 1.6).

We propose two approaches to the groups G attached to holonomic \mathcal{D} -modules on complex abelian varieties. The first one identifies them as the structure groups of certain principal bundles arising from the results of Schnell on the Fourier-Mukai transform [43]. As a consequence we will see that the group G/G° of connected components is a finite abelian group, dual to a finite group of points on the abelian variety (theorem 1.3). This extends a result of Weissauer for perverse sheaves [46] to irregular holonomic \mathcal{D} -modules and shows that the main task is to understand the connected component $G^\circ \subseteq G$. In the most interesting case of semisimple holonomic \mathcal{D} -modules this is a reductive group; its structure is governed by its root system, so we should look for multiplicative subgroups inside G . This is the second and main approach in this paper. Using classical results of Kashiwara from microlocal analysis [17], we give a construction of multiplicative subgroups and their normalizers in terms of characteristic cycles which will imply the relation between Weyl groups and monodromy groups mentioned above. This also explains the occurrence of minuscule representations in the case of smooth subvarieties, and it gives a powerful new way to determine the arising groups which covers all previously known examples in a uniform way (theorem 2.1).

With some control on the groups, one may then use representation theory as a tool to study subvarieties of abelian varieties. We illustrate this with two simple applications: Firstly we discuss a Tannakian obstruction for a subvariety to be a sum of small-dimensional subvarieties (proposition 2.3); this gives an easy way to see that the theta divisor on the intermediate Jacobian of a smooth cubic threefold is not a sum of curves, a special case of the beautiful result by Schreieder [44]. As a second application we prove a Torelli theorem which recovers a smooth subvariety from the corresponding group and its representation (corollary 2.5); this generalizes the classical Torelli theorems for curves and Fano surfaces of cubic threefolds, and it explains the relevance of the Tannakian setup for moduli problems.

Acknowledgements. I would like to thank Claude Sabbah for many enlightening discussions on \mathcal{D} -modules and for his kind hospitality at the École Polytechnique. I am indebted to Christian Schnell for sharing with me his analytic conjectures on the Fourier-Mukai transform that motivated the algebraic approach taken here, and for his suggestion to test the microlocalization functor on hyperelliptic Jacobians. My thanks also go to the organizers of the Conference on \mathcal{D} -modules and Singularities at Padova and those of the Séminaire de Géométrie Algébrique at Jussieu for the opportunity to present preliminary versions of some results in this paper, to Takuro Mochizuki and Will Sawin for their comments, and to the members and staff of the Centre de Mathématiques Laurent Schwartz at the École Polytechnique for a very pleasant research environment. This work has been funded by the DFG research grant *Holomome \mathcal{D} -Moduln auf abelschen Varietäten*.

1. BASIC STRUCTURE RESULTS

We now describe in more detail the basic results of this paper for the structure of Tannaka groups. Some geometric applications will be discussed in section 2 while the proofs are mostly deferred to sections 3 – 5.

1.a. The Tannakian setting. We apply the Tannakian constructions of [26] [21] in the following situation. For a complex abelian variety A we consider the abelian category $M_A = \text{Hol}(\mathcal{D}_A)$ of holonomic right modules for the sheaf \mathcal{D}_A of algebraic differential operators, and we denote by $D_A = D_{hol}^b(\mathcal{D}_A)$ the derived category of bounded algebraic \mathcal{D}_A -module complexes with holonomic cohomology sheaves. On the derived category the addition morphism $a : A \times A \rightarrow A$ defines a convolution product

$$\mathcal{M}_1 * \mathcal{M}_2 = a_*(\mathcal{M}_1 \boxtimes \mathcal{M}_2),$$

and D_A with this convolution product naturally becomes a *tensor category*, i.e. a symmetric monoidal \mathbb{C} -linear category in the sense of [29, sect. VII.7]. The abelian subcategory M_A is not stable under convolution, so we take a quotient category by negligible objects as follows [26]. Recall that for any holonomic module $\mathcal{M} \in M_A$, the de Rham complex

$$\text{DR}(\mathcal{M}) = \left[\cdots \rightarrow \mathcal{M} \otimes_{\mathcal{O}_A} \wedge^3 \mathcal{T}_A \rightarrow \mathcal{M} \otimes_{\mathcal{O}_A} \wedge^2 \mathcal{T}_A \rightarrow \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{T}_A \right]$$

is a perverse sheaf. Hence the Euler characteristic of its hypercohomology is always nonnegative by Kashiwara's index theorem [12, cor. 1.4]. We say \mathcal{M} is *negligible* if this Euler characteristic is zero. More generally, we say that an object of D_A is negligible if all its cohomology sheaves are so. The generic vanishing theorem and the classification of simple negligible modules in [26, prop. 10.1] or in [43, cor. 5.2] imply as in [13]:

- (1) The negligible objects form a Serre subcategory $S_A \subset M_A$ respectively a thick subcategory $T_A \subset D_A$.
- (2) The Verdier quotient $D(A) = D_A/T_A$ is a triangulated tensor category with a t -structure whose core is stable under convolution, and this core is naturally equivalent to the abelian quotient category $M(A) = M_A/S_A$.
- (3) This endows $M(A)$ with the structure of a rigid abelian tensor category all of whose finitely generated tensor subcategories are neutral Tannakian.

By a *finitely generated tensor subcategory* we mean the smallest rigid abelian tensor subcategory containing a given object $\mathcal{M} \in M(A)$, denoted $\langle \mathcal{M} \rangle \subset M(A)$ in what follows. A *neutral Tannakian category* is a rigid abelian tensor category whose unit object $\mathbf{1}$ has $\text{End}(\mathbf{1}) = \mathbb{C}$ and which admits a *fibre functor*, i.e. a faithful \mathbb{C} -linear exact tensor functor to the category $\text{Vect}(\mathbb{C})$ of finite dimensional complex vector spaces. Any fibre functor induces an equivalence

$$\omega : \langle \mathcal{M} \rangle \xrightarrow{\sim} \text{Rep}(G)$$

with the abelian tensor category of finite dimensional algebraic representations of the linear algebraic group $G = \text{Aut}(\omega|_{\langle \mathcal{M} \rangle})$ of tensor automorphisms of the fibre functor. Up to isomorphism this group is determined uniquely by the module \mathcal{M} since any two fibre functors on a neutral Tannakian category over an algebraically closed field are isomorphic [10, th. 3.2(b)]. So by abuse of notation we will simply write $G = G(\mathcal{M})$ when we only care about isomorphism types.

1.b. The Fourier-Mukai transform. In [26] fibre functors for perverse sheaves are obtained by taking the hypercohomology of the tensor product with a generic local system of rank one. Such local systems correspond to pairs (\mathcal{L}, ∇) where \mathcal{L} is a line bundle on A and $\nabla : \mathcal{L} \rightarrow \Omega_A^1 \otimes \mathcal{L}$ is a flat connection. We may consider all these fibre functors on an equal footing by working over the moduli space A^\natural of pairs (\mathcal{L}, ∇) . This moduli space is a smooth quasi-projective variety which is a torsor over the dual abelian variety via the map $A^\natural \rightarrow \hat{A} = \text{Pic}^\circ(A)$ that forgets the connection, see [43, sect. 9] and the references given there. On $A \times A^\natural$ the pullback of the Poincaré bundle has a universal relative flat connection. Using this as an integral kernel, Laumon [27] and Rothstein [37] have introduced the Fourier-Mukai transform

$$\text{FM} : D_{\text{coh}}^b(\mathcal{D}_A) \xrightarrow{\sim} D_{\text{coh}}^b(\mathcal{O}_{A^\natural})$$

from the derived category of bounded complexes of coherent algebraic \mathcal{D}_A -modules to the one of coherent algebraic \mathcal{O}_{A^\natural} -modules, and they have shown that this is an equivalence of categories. The vanishing theorem of Schnell [43] says that while the functor FM is not exact with respect to the standard t -structures, its restriction to the subcategory $D_{\text{hol}}^b(\mathcal{D}_A) \subset D_{\text{coh}}^b(\mathcal{D}_A)$ becomes an exact functor when $D_{\text{coh}}^b(\mathcal{O}_{A^\natural})$ is equipped with a certain perverse coherent t -structure [2] [19]. In particular, for any $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ there is an open subset $U \subseteq A^\natural$ with complement of codimension at least two such that

$$(1.1) \quad \mathcal{H}^i(\text{FM}(\mathcal{M}))|_U \text{ is } \begin{cases} \text{locally free} & \text{for } i = 0, \\ \text{zero} & \text{for } i \neq 0. \end{cases}$$

In fact $U \subseteq A^\natural$ can be taken to be the complement of a finite union of translates of proper linear subvarieties, where by a *linear subvariety* we mean the image of an embedding $B^\natural \hookrightarrow A^\natural$ given by an epimorphism $A \twoheadrightarrow B$ of abelian varieties. After removing finitely many further translates of proper linear subvarieties, we may assume all the subquotients of \mathcal{M} also have the property (1.1) on our chosen open subset. Under this assumption we may replace the tensor category $\langle \mathcal{M} \rangle \subset \text{M}(A)$ by an equivalent subcategory of modules all of whose negligible subquotients have their Fourier-Mukai transform supported outside U so that the fibre functors in [26] are given by the fibres

$$\omega_u = \mathcal{H}^0(\text{FM}(-))(u) : \langle \mathcal{M} \rangle \rightarrow \text{Vect}(\mathbb{C}) \quad \text{at } u \in U(\mathbb{C}).$$

This leads to the following interpretation of our Tannaka groups (see section 3.b):

Theorem 1.2. *In the above situation, the vector bundle $\mathcal{E} = \mathcal{H}^0(\text{FM}(\mathcal{M}))|_U$ is induced by an algebraic principal bundle whose structure group is isomorphic to the group $G(\mathcal{M}) = \text{Aut}(\omega_u|_{\langle \mathcal{M} \rangle})$ for any $u \in U(\mathbb{C})$. If \mathcal{M} is semisimple, then $G(\mathcal{M})$ is the unique minimal reductive structure group underlying this vector bundle.*

Here the minimality means that any other reduction of the vector bundle \mathcal{E} to an algebraic principal bundle with a reductive structure group is induced from the given one via an embedding of groups. In particular, if such a minimal reductive reduction exists at all, it is unique up to isomorphism; the existence has been shown in [4, th. 2.1] for vector bundles on any variety U with $H^0(U, \mathcal{O}) = 1$. So the Tannaka groups for semisimple holonomic \mathcal{D}_A -modules are determined uniquely by the Fourier-Mukai transform on the open dense subset $U \subseteq A^\natural$ with complement of codimension at least two, as should be expected from the reconstruction result

in [43, cor. 21.3]. We will use this in section 3.c for a simple proof of the following almost connectedness property which extends the result for perverse sheaves in [46] to irregular holonomic modules $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$.

Theorem 1.3. *The group of connected components of $G = G(\mathcal{M})$ is naturally a quotient $\pi_1(\hat{A}, 0) \rightarrow G/G^\circ$ of the fundamental group of the dual abelian variety \hat{A} .*

In particular, any object of the full subcategory $\text{Rep}(G/G^\circ) \subseteq \text{Rep}(G)$ is a sum of characters $\chi_a : \pi_1(\hat{A}, 0) \rightarrow G/G^\circ \rightarrow \mathbb{C}^*$ of finite order. The corresponding flat line bundles are parametrized by certain torsion points $a \in A = \text{Pic}^\circ(\hat{A})$, and if δ_a denotes the Dirac module supported on such a point, the associated line bundle is the Fourier-Mukai transform $\text{FM}(\delta_a)$. So the group of components is the Cartier dual

$$G/G^\circ = \text{Hom}(K, \mathbb{G}_m)$$

of the finite group

$$K = \{a \in A(\mathbb{C}) \mid \delta_a \in \langle \mathcal{M} \rangle \text{ and } n \cdot a = 0 \text{ for some } n \in \mathbb{N}\}$$

of torsion points whose Dirac modules occur in the rigid abelian tensor category generated by \mathcal{M} . For reference we also include in section 3.c the following formal consequence which is obtained by the Mackey-type arguments from [46].

Corollary 1.4. *With notations as above, let $\mathcal{N} = p_!(\mathcal{M})$ where $p : A \rightarrow B$ is an isogeny of abelian varieties. Then $G(\mathcal{N})$ is the subgroup of finite index in G given by*

$$G(\mathcal{N})/G^\circ = \text{Hom}(K/K \cap \ker(p), \mathbb{G}_m) \subseteq G/G^\circ = \text{Hom}(K, \mathbb{G}_m).$$

So the group of connected components of our Tannaka groups is easily controlled via finite subgroups of points on the abelian variety, and the main task is to study the connected component $G^\circ \subseteq G$. For this we propose a different class of fibre functors that are defined by microlocal constructions.

1.c. Characteristic cycles and the Gauss map. It has been observed in [23] that in all known examples the arising Tannaka groups are related to the geometry of conormal bundles. To explain this, consider for $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ the characteristic cycle

$$\text{CC}(\mathcal{M}) = \sum_{\Lambda} m_{\Lambda}(\mathcal{M}) \cdot \Lambda \quad \text{with} \quad m_{\Lambda}(\mathcal{M}) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where $\Lambda \subset T^*A$ runs through all irreducible closed conic Lagrangian subvarieties of the cotangent bundle [16, sect. 2.2]. Any such Λ is the closure of the conormal bundle to the smooth locus of some closed subvariety of A . Now A has the trivial cotangent bundle

$$T^*A = A \times V \quad \text{with fibre} \quad V = H^0(A, \Omega_A^1),$$

so we may generalize the usual Gauss map for a smooth divisor by considering the projection $\gamma_{\Lambda} : \Lambda \subset A \times V \rightarrow V$ onto the fibre. This *generalized Gauss map* is generically finite and we denote by $d_{\Lambda} \in \mathbb{N}_0$ its generic degree. We put $d_{\Lambda} = 0$ if the Gauss map is not dominant, and we then say Λ is *degenerate*. The index formula in [12, th. 1.3 and prop. 2.2] says

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} H^i(A, \text{DR}(\mathcal{M})) = \sum_{\Lambda} m_{\Lambda}(\mathcal{M}) \cdot d_{\Lambda}.$$

In particular, a module is negligible iff its characteristic cycle is a sum of degenerate Lagrangian subvarieties. The main goal of this paper is to categorify the index formula: We naturally attach to any \mathcal{M} a local system of rank $m_\Lambda(\mathcal{M})$ on an open dense subset of each nondegenerate irreducible component Λ of the characteristic variety $\text{Char}(\mathcal{M}) = \text{Supp}(\text{CC}(\mathcal{M})) \subset T^*A$.

There is a classical such construction, the *second microlocalization* [14, sect. 6], but it gives *twisted local systems* [8] and the passage from those to untwisted ones is not canonical. For a construction that works well with the Tannakian formalism we must remove the twist in a way that is compatible with convolution products in a suitable sense. We do this in two steps: First we pass to microdifferential modules on the cotangent bundle (sections 5.a and 5.b), then we compare these with a class of simple microdifferential modules that behave well with respect to convolution products (sections 5.c and 5.d). Each of our simple modules will only be defined on the locus where the corresponding Gauss map is a finite étale cover, and this locus cannot be fixed in advance if we want to consider convolution products. So in section 4 we introduce an abelian tensor category $\text{LS}(A, \eta)$ of germs of local systems near the generic point $\eta \in V$ for which our untwisted version of the second microlocalization will result in a tensor functor $\omega_\eta : \text{M}(A) \rightarrow \text{LS}(A, \eta)$.

To see what this means for the group $G(\mathcal{M})$ attached to a module $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$, consider the Gauss map

$$\gamma : \text{Char}(\mathcal{M}) \subset A \times V \rightarrow V$$

on the characteristic variety. On some Zariski open dense subset this map restricts to a finite étale cover, and we define its monodromy group to be the automorphism group $\text{Gal}(\gamma)$ of its Galois hull. For a very general point $u \in V(\mathbb{C})$, i.e. all u outside countably many proper closed subvarieties, we will specialize the functor ω_η to a fibre functor ω_u that factors as follows:

$$\begin{array}{ccc} \langle \mathcal{M} \rangle & \xrightarrow{\omega_u} & \text{Vect}(\mathbb{C}) \\ & \searrow \exists & \nearrow \\ & \text{Vect}_{X_u}(\mathbb{C}) & \end{array}$$

Here $\text{Vect}_{X_u}(\mathbb{C})$ is the category of finite dimensional vector spaces with a grading by the group

$$X_u = \langle a \in A(\mathbb{C}) \mid (a, u) \in \text{Char}(\mathcal{M}) \rangle \subset A(\mathbb{C})$$

generated by the finitely many points in the fibre of the Gauss map. We then get the following result, where for a multiplicative subgroup $T \hookrightarrow G$ of an algebraic group G we denote by $W(G, T) = N_G(T)/Z_G(T)$ the quotient of its normalizer by its centralizer; for a maximal torus this is the usual Weyl group.

Theorem 1.5. *Fix \mathcal{M} as above. Putting $G_u = \text{Aut}(\omega_u \mid \langle \mathcal{M} \rangle)$ for very general u , we have natural embeddings*

$$T_u = \text{Hom}(X_u, \mathbb{G}_m) \hookrightarrow G_u \quad \text{and} \quad \text{Gal}(\gamma) \hookrightarrow W_u = W(G_u, T_u).$$

Note that the monodromy action on the fibre of the Gauss map induces an action of $\text{Gal}(\gamma)$ on the Cartier dual $T_u = \text{Hom}(X_u, \mathbb{G}_m) \hookrightarrow G_u$. The proof of theorem 4.3 will more precisely show that this monodromy action is induced by the conjugation action $W_u \rightarrow \text{Aut}(T_u)$ via the above embeddings.

1.d. Weight spaces and minuscule representations. The above is most useful if the arising multiplicative subgroups are as large as possible. While for arbitrary modules $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ it seems hard to control this, there is a favorable class of geometric examples where we can say more.

Let us say that a module $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ is *generically multiplicity-free* if any nondegenerate irreducible component of its characteristic cycle has multiplicity one and if none of these components is mapped to itself or to another component via a translation by a torsion point in $A(\mathbb{C})$. For example, if $i : Y \hookrightarrow A$ is a smooth irreducible subvariety that is not invariant under any translation, then $\delta_Y = i_*(\omega_Y)$ is generically multiplicity-free. Note that this applies in particular to any smooth subvariety $Y \subset A$ which is a summand of a divisor defining a principal polarization, such as a smooth curve inside its Jacobian variety, the Fano surface of lines on a smooth cubic threefold inside the intermediate Jacobian, or any smooth theta divisor. If G is an algebraic group, let

$$W(G) = W(G^\circ, T)$$

denote the Weyl group of its connected component with respect to any maximal torus $T \subseteq G$. For any representation $U \in \text{Rep}(G)$ the restriction $U|_T \in \text{Rep}(T)$ splits into a sum of one-dimensional representations which we call the *weights* of the representation, and these weights are permuted by the Weyl group. Up to isomorphism the Weyl group and its permutation action on the weights do not depend on the chosen maximal torus. In our case they contain all monodromy data of the Gauss map as conjectured in [23]:

Corollary 1.6. *If \mathcal{M} is generically multiplicity-free, then with notations as above we have an embedding*

$$\text{Gal}(\gamma) \hookrightarrow W(G_u).$$

Moreover, there is a bijection between the fibre $\gamma^{-1}(u) \subset \text{Char}(\mathcal{M})$ and the weights in $\omega_u(\mathcal{M}) \in \text{Rep}(G_u)$ so that the monodromy and Weyl group actions match.

Proof. One easily checks that the direct image of a generically multiplicity free module under an isogeny remains generically multiplicity free. So by corollary 1.4 we may assume that

- the group $G = G_u$ is connected, and
- the subgroup $X = X_u \subset A(\mathbb{C})$ is torsion-free.

The second condition means that the multiplicative subgroup $T = \text{Hom}(X, \mathbb{G}_m)$ in theorem 1.5 is a torus, and by generic multiplicity-freeness the restriction of the faithful representation $W = \omega_u(\mathcal{M}) \in \text{Rep}(G)$ to this torus decomposes into a sum of pairwise distinct characters. Since the centralizer $Z_G(T)$ preserves this decomposition, it is contained in a subgroup of diagonal matrices in $GL(W)$. Its connected component $Z_G(T)^\circ$ is therefore a subtorus of G , hence a maximal torus because any other subtorus containing T also lies in the centralizer. The image of the fundamental group under the monodromy representation normalizes this maximal torus since $N_G(T) \subseteq N_G(Z_G(T)) \subseteq N_G(Z_G(T)^\circ)$. \square

To illustrate the strength of this result, recall that an irreducible representation of a connected algebraic group is called *minuscule* if all its weights with respect to a maximal torus are in the same Weyl group orbit. Such representations are very

special [15]. For the Dynkin types of the simple Lie algebras, the only examples are the fundamental representations of the following dimensions:

A_n	B_n	C_n	D_n	E_6	E_7
$\binom{n+1}{k}$ for $1 \leq k \leq n$	2^n	$2n$	$2n, 2^{n-1}$	27	56

In particular, there are no minuscule representations for the types E_8, F_4, G_2 , so we may exclude these types in many cases:

Corollary 1.7. *Let $\Lambda \subset T^*A$ be an irreducible Lagrangian subvariety not stable under any translation by a point of $A(\mathbb{C})$. If $\mathcal{M} \in \text{Hol}(A)$ has the characteristic cycle $\text{CC}(\mathcal{M}) \equiv \Lambda$ modulo a linear combination of degenerate subvarieties, then the corresponding representation of $G(\mathcal{M})^\circ \subset G(\mathcal{M})$ is minuscule.*

Proof. The irreducibility of Λ means that the monodromy group acts transitively on the fibre of the Gauss map, hence a fortiori the Weyl group of the connected component acts transitively on the weights in the corresponding representation. \square

2. SOME GEOMETRIC APPLICATIONS

In this section we illustrate the above results with some simple applications: A general criterion to determine Tannaka groups, an obstruction for a subvariety to be a sum of small-dimensional varieties, and a Tannakian Torelli theorem.

2.a. Examples with big monodromy. The study of the Gauss map gives an efficient way to determine Tannaka groups if we have a priori upper bounds on these groups and corresponding lower bounds on the monodromy of the Gauss map.

Upper bounds on Tannaka groups come from invariants in tensor powers. For example, if $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ has *trivial determinant* in the sense that its top exterior convolution power is the Dirac module $\mathbf{1} = \delta_0$ at the origin, then $G(\mathcal{M})$ embeds in the corresponding special linear group. This always holds after translation by a point in $A(\mathbb{C})$ since any one-dimensional representation in our Tannakian categories is given by a Dirac module [26, prop. 10.1]. Similarly, we say \mathcal{M} is *self-dual* if it is isomorphic to

$$\mathcal{M}^\vee = (-id_A)^* D(\mathcal{M}),$$

where $D = R\mathcal{H}om_{\mathcal{D}_A}(-, \mathcal{D}_A \otimes \omega_A)[\dim A]$ denotes the usual duality functor. Then the associated representation is isomorphic to its dual, the characteristic variety is stable under $-id_A$, and the monodromy of the Gauss map $\gamma : \text{Char}(\mathcal{M}) \rightarrow V$ is a subgroup

$$\text{Gal}(\gamma) \hookrightarrow (\pm 1)^{d/2} \rtimes \mathfrak{S}_{d/2} \quad \text{for } d = \deg(\gamma).$$

As a final upper bound from invariants in tensor powers, we say that \mathcal{M} admits a *cubic form* if there is a nontrivial morphism $\mathcal{M} * \mathcal{M} * \mathcal{M} \rightarrow \mathbf{1}$. To formulate lower bounds on monodromy groups, recall that up to duality the symmetric group \mathfrak{S}_d has a unique irreducible complex representation of dimension $d - 1$. We say that a subgroup of \mathfrak{S}_d is *irreducible* if it acts irreducibly on this representation; such subgroups are classified in [40]. Similarly, the group $H = (\pm 1)^{d/2} \rtimes \mathfrak{S}_{d/2}$ has a natural faithful $d/2$ -dimensional irreducible complex representation, and we say that a subgroup of H is *irreducible* if it acts irreducibly on this representation.

With this terminology, all previously known examples of Tannaka groups are covered by the following result:

Theorem 2.1. *Let $G = G(\mathcal{M})$, where \mathcal{M} is a simple generically multiplicity-free module with trivial determinant and Gauss map $\gamma : \text{Char}(\mathcal{M}) \rightarrow V$ of degree d .*

- (1) *If $\text{Gal}(\gamma)$ is an irreducible subgroup of the symmetric group \mathfrak{S}_d via the monodromy operation, then $G \simeq \text{Sl}_d(\mathbb{C})$.*
- (2) *If \mathcal{M} is self-dual and $\text{Gal}(\gamma)$ is an irreducible subgroup of $(\pm 1)^{d/2} \rtimes \mathfrak{S}_{d/2}$ via the monodromy operation, then*

$$G \simeq \text{SO}_d(\mathbb{C}) \quad \text{or} \quad G \simeq \text{Sp}_d(\mathbb{C}).$$

- (3) *If \mathcal{M} admits a cubic form, $d = 27$ and $\text{Gal}(\gamma) \simeq W(E_6)$, then $G \simeq E_6(\mathbb{C})$.*

In all three cases the subgroup $T_u \subset G$ is a maximal torus for very general u .

Proof. The triviality of the determinant implies $G \hookrightarrow \text{Sl}_d(\mathbb{C})$, and in the self-dual case $G \hookrightarrow \text{SO}_d(\mathbb{C})$ or $G \hookrightarrow \text{Sp}_d(\mathbb{C})$ since every irreducible self-dual representation is orthogonal or symplectic. Furthermore, the transitivity of the monodromy shows that the characteristic cycle $\text{CC}(\mathcal{M})$ is irreducible up to a linear combination of degenerate subvarieties, so by corollary 1.7 the representation of $G^\circ \subseteq G$ defined by \mathcal{M} is minuscule. Together with the given degree $d = 27$ and the existence of a cubic form, this easily implies $G \simeq E_6(\mathbb{C})$ in case 3. So it suffices to show that for very general u the rank of the finitely generated abelian group $X = X_u \subset A(\mathbb{C})$ is at least $d - 1$ in case 1, at least $d/2$ in case 2 and at least six in case 3. But this rank is the dimension of the representation

$$X_{\mathbb{C}} = X \otimes_{\mathbb{Z}} \mathbb{C} \in \text{Rep}(\text{Gal}(\gamma)),$$

and the latter is faithful because by generic multiplicity-freeness no two points in a general fibre of the Gauss map differ by a torsion point. Hence in cases 1 and 2 the claim follows from the observation that this representation is a quotient of the natural representation of dimension $d - 1$ resp. $d/2$, on which $\text{Gal}(\gamma)$ acts irreducibly by assumption. In case 3 the claim follows from the fact that the smallest dimension of a faithful representation of $W(E_6)$ is six, see [7, p. 27]. \square

In [23, th. 9] the monodromy of the Gauss map has been listed for the known examples of Tannaka groups, and the above theorem covers all the examples. More specifically it applies to the intersection cohomology module $\mathcal{M} = \delta_S$ on a suitable translate $S \hookrightarrow A$ of

- any smooth projective curve inside its Jacobian variety [45] [24],
- the theta divisor of a generic principally polarized abelian variety [25],
- the Fano surfaces of lines on a smooth cubic threefold, embedded into the intermediate Jacobian [23].

Let us take a closer look at the last example:

Example 2.2. Let $S \subset A$ be a translate of the Fano surface on the intermediate Jacobian of a smooth cubic threefold. The sum map $a : S \times S \times S \rightarrow A$ collapses the incidence variety of coplanar triples of lines on the threefold to a single point [6], and after a translation we may assume this point is the origin. From a look at the fibre dimensions and the decomposition theorem for perverse sheaves one may then show that $\delta_0 \hookrightarrow \delta_S * \delta_S * \delta_S$, and this gives a cubic form on $\mathcal{M} = \delta_S$. Our interpretation of weights also explains why in this example the characteristic cycle

of the intersection cohomology module on the theta divisor contains the fibre over the origin with multiplicity $m = 6$ as noted in loc. cit. rem. 7(b). Indeed, we know from loc. cit. that for the intermediate Jacobian the theta divisor corresponds to the adjoint representation of the group $G(\delta_S)$ on its Lie algebra, so the multiplicity of the zero weight is equal to the rank of a maximal torus.

2.b. Sums of subvarieties. The Tannakian setup allows to apply representation theory to see whether a given subvariety $W \subset A$ can be written as a sum of small-dimensional varieties. Leaving a closer study to a future work, we here only illustrate the basic idea with a simple example. Let us say that an irreducible subvariety $W \subset A$ is *nondegenerate* if the Gauss map $\gamma : \text{Char}(\delta_W) \rightarrow V$ for the intersection cohomology module δ_W is dominant so that the corresponding Tannaka group $G(\delta_W) \neq \{1\}$ is not the trivial group.

Proposition 2.3. *Let $W \subset A$ be an irreducible nondegenerate subvariety, and suppose $G(\delta_W)$ is almost simple in the sense that the adjoint representation of its connected component on its Lie algebra is irreducible. If $W = Y_1 + \cdots + Y_n$ is a sum of subvarieties $Y_i \subset A$, then*

$$\dim(Y_i) \geq \frac{\dim(\text{Supp}(\mathcal{M}))}{2} \quad \text{for some } i,$$

where $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ is the simple module defined by the adjoint representation.

Proof. If W can be written as such a sum, the decomposition theorem gives an inclusion

$$\delta_W \hookrightarrow \delta_{Y_1} * \cdots * \delta_{Y_n} \in \langle \delta_Y \rangle = \langle \delta_{Y_1} \oplus \cdots \oplus \delta_{Y_n} \rangle$$

in the Tannakian category generated by the intersection cohomology module on the union $Y = Y_1 \cup \cdots \cup Y_n$. By the Tannakian formalism [10, prop. 2.21] such an inclusion defines an epimorphism

$$p : G(\delta_Y) \twoheadrightarrow G(\delta_W)$$

of algebraic groups. Passing to direct images under an isogeny of abelian varieties we may assume by corollary 1.4 that the reductive groups $G(\delta_Y)$ and $G(\delta_W)$ are connected. In fact we can assume them to be semisimple because we can replace each Y_i and W by a translate so that the determinant of the corresponding faithful irreducible representations is trivial. Now any connected semisimple algebraic group is isogenous to its universal covering group. Let us denote by $q : \tilde{G}(\delta_Y) \twoheadrightarrow G(\delta_Y)$ and by $\tilde{G}(\delta_W) \twoheadrightarrow G(\delta_W)$ these universal covering groups. Then by the theory of connected semisimple groups there exists a section ι of p up to isogeny as indicated in the diagram

$$\begin{array}{ccc} \tilde{G}(\delta_Y) & \xrightarrow{q} & G(\delta_Y) \\ \downarrow & & \downarrow p \\ \tilde{G}(\delta_W) & \xrightarrow{\quad} & G(\delta_W) \end{array} \quad \begin{array}{c} \nearrow \exists \iota \\ \vdots \end{array}$$

and the image of this section commutes with the kernel $K = \ker(p \circ q)$. So we get an isomorphism

$$G = K \times \tilde{G}(\delta_W) \xrightarrow{\sim} \tilde{G}(\delta_Y),$$

and in what follows we consider all occurring intersection cohomology modules as representations of G via the fully faithful embedding $\omega : \langle \delta_Y \rangle \hookrightarrow \text{Rep}(G)$.

By assumption the adjoint representation $\mathbf{1} \boxtimes Ad \in \text{Rep}(G)$ on the Lie algebra of $G(\delta_W)$ is irreducible, so this Lie algebra is simple. It then follows that any nontrivial representation of $G(\delta_W)$ is almost faithful in the sense that its kernel is finite. Since the tensor product of any almost faithful representation with its dual contains the adjoint representation, we therefore obtain an inclusion

$$\omega(\mathcal{M}) = \mathbf{1} \boxtimes Ad \hookrightarrow \omega(\delta_Y) \otimes \omega(\delta_Y)^\vee = \omega(\delta_Y * \delta_{-Y})$$

which geometrically corresponds to an inclusion $\mathcal{M} \hookrightarrow \delta_Y * \delta_{-Y}$ in $\text{Hol}(\mathcal{D}_A)$. But then $\text{Supp}(\mathcal{M}) \hookrightarrow Y - Y \subset A$. \square

An interesting special case is the question whether a given subvariety W can be written as a sum of curves. A result of Schreieder [44, cor. 3] says that the theta divisor $\Theta \subset A$ on an indecomposable principally polarized abelian variety is a sum of curves only if (A, Θ) is the Jacobian of a smooth projective curve. We do not attempt to give a new proof of this but illustrate our Tannakian method with the following special case:

Corollary 2.4. *The theta divisor on the intermediate Jacobian A of a smooth cubic threefold is not dominated by a product of curves.*

Proof. For the intermediate Jacobian the intersection cohomology module δ_Θ of a suitable translate $\Theta \subset A$ of the theta divisor corresponds on the Tannakian side to the adjoint representation of $G(\delta_\Theta)$ by [23], so the above proposition applies. \square

2.c. A Torelli theorem for subvarieties. Our last application is a Tannakian Torelli theorem that recovers a smooth subvariety $Y \hookrightarrow A$ from the group $G(\delta_Y)$ and the corresponding representation, provided that the relation of these data to a chosen simple reference module $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ is specified. This contains the Torelli theorems for curves and for Fano surfaces of smooth cubic threefolds as special cases by taking $\mathcal{M} = \delta_\Theta$ to be the intersection cohomology module of the theta divisor on the Albanese variety; it may be instructive to compare the argument with the role of the Gauss map in classical proofs such as the one by Andreotti [1]. To rigidify the isomorphisms between Tannaka groups, we fix a very general point $u \in V(\mathbb{C})$ and put $G_u(\mathcal{M}) = \text{Aut}(\omega_u|_{\langle \mathcal{M} \rangle})$ as in theorem 1.5.

Corollary 2.5. *Fix a simple module $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$. For $i = 1, 2$, let $Y_i \subset A$ be subvarieties such that*

- $\text{CC}(\delta_{Y_i})$ is irreducible and not stable under any translation,
- $\mathcal{M} \in \langle \delta_{Y_i} \rangle$ and the induced map $p_i : G_u(\delta_{Y_i}) \rightarrow G_u(\mathcal{M})$ is an isogeny,
- there is an isomorphism φ for which the diagram

$$\begin{array}{ccc} G_u(\delta_{Y_1}) & \xrightarrow[\varphi]{\sim} & G_u(\delta_{Y_2}) \\ & \searrow p_1 \quad \swarrow p_2 & \\ & G_u(\mathcal{M}) & \end{array}$$

commutes and which satisfies $\varphi^(\omega_u(\delta_{Y_2})) \simeq \omega_u(\delta_{Y_1})$.*

Then $Y_1 = Y_2 + a$ for some n -torsion point $a \in A(\mathbb{C})$, where $n = \deg(p_1) = \deg(p_2)$.

Proof. Put $G = G_u(\mathcal{M})$, $G_i = G_u(\delta_{Y_i})$, and let $T \subseteq G$ and $T_i \subseteq G_i$ be the subtori which are the connected components of the multiplicative subgroups in

theorem 1.5. By assumption we have an inclusion $\mathcal{M} \in \langle \delta_{Y_i} \rangle$. So an application of Kashiwara's estimate for the characteristic variety of a direct image [18, th. 4.27] gives an inclusion

$$\langle a \in A(\mathbb{C}) \mid (a, u) \in \text{Char}(\mathcal{M}) \rangle \subseteq \langle a \in A(\mathbb{C}) \mid (a, u) \in \text{Char}(\delta_{Y_i}) \rangle$$

between the groups generated by the fibres of the corresponding Gauss maps. The Cartier dual of an inclusion is an epimorphism, so the isogeny $p_i : G_i \twoheadrightarrow G$ restricts to an isogeny $T_i \twoheadrightarrow T$ between the chosen subtori. Then by connectedness T_i coincides with the connected component of the preimage $p_i^{-1}(T) \subseteq G_i$, hence we have $\varphi(T_1) = T_2$ since $p_2 \circ \varphi = p_1$. The generic multiplicity freeness and the same centralizer argument as in the proof of corollary 1.6 furthermore show that each T_i lies in a unique maximal torus $Z_i \subseteq G_i$, and the uniqueness of these maximal tori implies that $\varphi(Z_1) = Z_2$ and that the maximal torus $Z = p_1(Z_1) = p_2(Z_2) \subseteq G$ is determined uniquely as well. So we have found a distinguished maximal torus in each of our groups such that p_1, p_2 and φ map these tori onto each other. We want to use the weight space decomposition with respect to these tori to identify suitable irreducible components in the characteristic varieties of convolution powers of δ_{Y_1} and δ_{Y_2} which suffice to recover Y_1 and Y_2 .

The kernel $\ker(p_i) \subseteq G_i$ is a central subgroup of order n , so by Schur's lemma it acts on the irreducible representation $U_i = \omega_u(\delta_{Y_i}) \in \text{Rep}(G_i)$ by multiplication with n -th roots of unity. This action induces the trivial action on the n -th tensor power, so via the fully faithful embedding $p_i^* : \text{Rep}(G) \hookrightarrow \text{Rep}(G_i)$ we may view the symmetric power

$$W_i = S^n(U_i)$$

as an object of $\text{Rep}(G)$, corresponding to a module $\mathcal{N}_i \in \langle \mathcal{M} \rangle$. Note that $\text{Char}(\mathcal{N}_i)$ has a unique irreducible component

$$\Delta_i \subseteq \text{Char}(\mathcal{N}_i) \subseteq \text{Char}(\delta_{Y_i} * \cdots * \delta_{Y_i})$$

that is the image of the diagonal under the addition map

$$\text{Char}(\delta_{Y_i}) \times_V \cdots \times_V \text{Char}(\delta_{Y_i}) \longrightarrow \text{Char}(\delta_{Y_i} * \cdots * \delta_{Y_i})$$

and hence surjects onto the subvariety $n \cdot Y_i = \{ny \mid y \in Y_i\} \subset A$. On the representation theoretic side this simply means that among all the weights in W_i for the maximal torus $Z \subseteq G$, there exists a unique Weyl group orbit consisting of weights which are n times a weight of U_i for the maximal torus $Z_i \subseteq G_i$. By corollary 1.6 this Weyl group orbit corresponds to a single orbit of the monodromy group of the Gauss map, hence to an irreducible component of $\text{Char}(\mathcal{N}_i)$.

Now by assumption we have an isomorphism $\varphi^*(U_2) \simeq U_1$ in $\text{Rep}(G_1)$, hence an isomorphism $W_2 \simeq W_1$ in $\text{Rep}(G)$, and under this latter isomorphism our Weyl group orbits correspond to each other because our maximal tori are compatible with p_1, p_2 and φ . Geometrically this gives an isomorphism $\mathcal{N}_2 \simeq \mathcal{N}_1$ in $\langle \mathcal{M} \rangle$ and shows that the diagonal components

$$\Delta_1, \Delta_2 \subseteq \text{Char}(\mathcal{N}_1) = \text{Char}(\mathcal{N}_2)$$

coincide. Projecting onto the abelian variety A we obtain that $n \cdot Y_1 = n \cdot Y_2$ and the claim easily follows. \square

The ambiguity of signs in the classical Torelli theorem is of course still present in the above formulation: While we may distinguish a nonhyperelliptic curve from its

negative, there is no preferred choice between them. On the Tannakian side, there is no preferred choice between a representation and its dual without a labelling for the weights — unless the representation coincides with its dual, which happens precisely for hyperelliptic curves.

3. THE FOURIER-MUKAI TRANSFORM

In this section we discuss an interpretation of our Tannaka groups as structure groups of certain principal bundles defined via the Fourier-Mukai transform. As an application we obtain that these groups are almost connected.

3.a. Algebraic principal bundles. Let G be a linear algebraic group and U an algebraic variety over the complex numbers. By a *principal bundle* with structure group G on U we mean an algebraic variety \mathcal{G} that is equipped with a surjective flat affine morphism $\mathcal{G} \rightarrow U$ and with a right action $m : \mathcal{G} \times G \rightarrow \mathcal{G}$ over U such that the diagram

$$\begin{array}{ccc} \mathcal{G} \times G & \xrightarrow{m} & \mathcal{G} \\ p \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & U \end{array}$$

is Cartesian, where p denotes the projection onto the first factor. For any algebraic representation $E \in \text{Rep}(G)$ we then get an associated vector bundle $\mathcal{E} = \mathcal{G} \times^G E$ by the usual contracted product. This gives a \mathbb{C} -linear exact tensor functor

$$\Phi = \Phi_{\mathcal{G}} : \text{Rep}(G) \longrightarrow \text{Coh}(\mathcal{O}_U)$$

to the category of coherent sheaves with the property that $\Phi(E)$ is a locally free sheaf of rank $\dim(E)$ for all $E \in \text{Rep}(G)$. Conversely it has been shown by Nori in [34, sect. 2] that any \mathbb{C} -linear exact tensor functor $\Phi : \text{Rep}(G) \rightarrow \text{Coh}(\mathcal{O}_U)$ with this latter property arises from a principal bundle in the above way and that up to isomorphism the bundle is determined uniquely by the functor. In the next section we will apply this to the group $G = G(\mathcal{M})$ attached to a module $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$, with Φ defined in terms of the Fourier-Mukai transform.

We say that an algebraic vector bundle has a *reduction* to a principal bundle \mathcal{G} if it lies in the essential image of the functor $\Phi_{\mathcal{G}}$. Any vector bundle of rank n has a reduction to a principal bundle with structure group $GL_n(\mathbb{C})$, but usually also many smaller ones. A reduction of a given vector bundle to a principal bundle \mathcal{H} is called a *natural reduction* if

- the structure group H of the principal bundle \mathcal{H} is reductive, and
- for any other reduction of the given vector bundle to a principal bundle \mathcal{G} with a reductive structure group G there exists an embedding $\iota : H \hookrightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Rep}(G) & \xrightarrow{\Phi_{\mathcal{G}}} & \text{Coh}(\mathcal{O}_U) \\ & \searrow \iota^* & \nearrow \Phi_{\mathcal{H}} \\ & \text{Rep}(H) & \end{array}$$

A result of Bogomolov [4, th. 2.1] says that on varieties U with $\dim H^0(U, \mathcal{O}_U) = 1$ any algebraic vector bundle has a unique such natural reduction, and for stable vector bundles on projective varieties the corresponding minimal reductive structure group is an algebraic analog of the holonomy of Chern connections in Riemannian geometry [3]. We will be guided by this analogy, but replace the stability condition by semisimplicity properties of our Tannakian categories.

3.b. The Fourier-Mukai transform. Let A^\natural be the moduli space of coherent line bundles with a flat connection on the abelian variety A as in section 1.b, and let p_1, p_2 be the two projections in the following diagram:

$$\begin{array}{ccc} & A \times A^\natural & \\ p_1 \swarrow & & \searrow p_2 \\ A & & A^\natural \end{array}$$

The pullback $\mathcal{P}_{|A^\natural}$ of the Poincaré bundle to the product $A \times A^\natural$ admits a universal relative flat connection

$$\nabla : \mathcal{P}_{|A^\natural} \longrightarrow \Omega_{A \times A^\natural / A^\natural}^1 \otimes_{\mathcal{O}_{A \times A^\natural}} \mathcal{P}_{|A^\natural}$$

and the Fourier-Mukai transform

$$\begin{aligned} \text{FM} : D_{coh}^b(\mathcal{D}_A) &\xrightarrow{\sim} D_{coh}^b(\mathcal{O}_{A^\natural}), \\ \mathcal{M} &\mapsto Rp_{2,*} \left(\text{DR}_{A \times A^\natural / A^\natural} (p_1^*(\mathcal{M}) \otimes (\mathcal{P}_{|A^\natural}, \nabla)) \right) \end{aligned}$$

is by definition the integral transform with this universal relative flat connection as its kernel [27] [37]. Note that in contrast to loc. cit. we use right modules. The relative de Rham complex of the right $\mathcal{D}_{A \times A^\natural / A^\natural}$ -module $\mathcal{N} = p_1^*(\mathcal{M}) \otimes (\mathcal{P}, \nabla)$ has the form

$$\text{DR}_{A \times A^\natural / A^\natural}(\mathcal{N}) = \left[\cdots \longrightarrow \mathcal{N} \otimes_{\mathcal{O}_{A \times A^\natural}} \wedge^2 \mathcal{T}_{A \times A^\natural / A^\natural} \longrightarrow \mathcal{N} \otimes_{\mathcal{O}_{A \times A^\natural}} \mathcal{T}_{A \times A^\natural / A^\natural} \right]$$

where $\mathcal{T}_{A \times A^\natural / A^\natural}$ denotes the relative tangent sheaf and where by convention we place the complex in nonpositive degrees so that it is relatively perverse. The reason for working with right rather than left modules are the signs for the commutativity constraints in our tensor categories:

Proposition 3.1. *The functor $\text{FM} : D_{coh}^b(\mathcal{D}_A) \longrightarrow D_{coh}^b(\mathcal{O}_{A^\natural})$ underlies a tensor functor with respect to the tensor structures given by the convolution product on the source and by the usual derived tensor product on the target.*

Proof. By [27, prop. 3.3.2] we have natural isomorphisms

$$\text{FM}(\mathcal{M}_1 * \mathcal{M}_2) \xrightarrow{\sim} \text{FM}(\mathcal{M}_1) \otimes_{\mathcal{O}_{A^\natural}}^L \text{FM}(\mathcal{M}_2),$$

and we claim that these are compatible with the associativity, commutativity and unit constraints. The crucial part is the commutativity constraint where we must see that the correct signs occur. This amounts to the compatibility of the relative de Rham functor for right modules with the action of the symmetric group on external tensor products and follows as in the absolute case [30, prop. 1.5]. \square

We now want to apply the generic vanishing property (1.1) from section 1.b to get a tensor functor with values in locally free sheaves on any finitely generated tensor subcategory of $M(A)$. However, by definition $M(A)$ is the quotient of $\text{Hol}(\mathcal{D}_A)$ by the Serre subcategory of *all* negligible modules, whereas for the construction of

principal bundles we want to work on a *fixed* open subset $U \subseteq A^\natural$ and only take the quotient by those negligible modules whose Fourier-Mukai transform vanishes on this given subset. So let $\text{Hol}(\mathcal{D}_A, U) \subset \text{Hol}(\mathcal{D}_A)$ be the full abelian subcategory of all holonomic modules with the property that the generic vanishing condition (1.1) holds on U for all their subquotients, and let $M(A, U)$ be the quotient of this full abelian subcategory by the Serre subcategory of all negligible modules contained in it. We have a commutative diagram

$$\begin{array}{ccc} \text{Hol}(\mathcal{D}_A, U) & \hookrightarrow & \text{Hol}(\mathcal{D}_A) \\ \downarrow & & \downarrow \\ M(A, U) & \xrightarrow{\exists! i} & M(A) \end{array}$$

where i comes from the universal property of the quotient $M(A)$ and is a fully faithful embedding by [26, lemma 12.3]. As in theorem 13.2 of loc. cit. the essential image of i is stable under the convolution product, and this endows $M(A, U)$ with a natural structure of a rigid abelian tensor category which is compatible with our previous constructions: The tensor subcategories generated by $\mathcal{M} \in \text{Hol}(A, U)$ in $M(A, U)$ and in $M(A)$ are naturally equivalent. Now we obtain a well-defined functor

$$\Phi : M(A, U) \longrightarrow \text{Coh}(\mathcal{O}_U), \quad \mathcal{M} \mapsto \mathcal{H}^0(\text{FM}(\mathcal{M}))|_U.$$

In the present terminology, theorem 1.2 takes the following form:

Theorem 3.2. *For any $\mathcal{M} \in \text{Hol}(\mathcal{D}_A, U)$, the vector bundle $\Phi(\mathcal{M}) \in \text{Coh}(\mathcal{O}_U)$ is induced by a principal bundle*

$$\mathcal{G} \longrightarrow U$$

whose structure group is isomorphic to the Tannaka group $G = G(\mathcal{M})$, and if \mathcal{M} is semisimple, then \mathcal{G} is a natural reduction for this vector bundle.

Proof. By the generic vanishing property (1.1) the functor Φ is exact and takes values in locally free sheaves, and it is a tensor functor by proposition 3.1. So the composite functor

$$\text{Rep}(G) \xrightarrow{\sim} \langle \mathcal{M} \rangle \subset M(A, U) \xrightarrow{\Phi} \text{Coh}(\mathcal{O}_U)$$

satisfies Nori's properties of section 3.a and thus comes from a principal bundle \mathcal{G} as claimed in the first part of the theorem. For the second part notice that

$$\dim H^0(U, \mathcal{O}_U) = \dim H^0(A^\natural, \mathcal{O}_{A^\natural}) = 1,$$

where the first identity holds since we assumed that the complement of $U \subseteq A^\natural$ is of codimension at least two, and the second is shown in [27, th. 2.4.1]; we are indebted to Christian Schnell for this observation. So by the result of Bogomolov [4, th. 2.1], the vector bundle $\Phi(\mathcal{M})$ has a natural reduction \mathcal{H} . Let H be the corresponding minimal reductive structure group. If \mathcal{M} is a semisimple module, the Tannakian category generated by it is semisimple as well, so the Tannaka group $G = G(\mathcal{M})$ is reductive. Hence by the minimality property of a natural reduction there exists an

embedding $\iota : H \hookrightarrow G$ such that the diagram

$$\begin{array}{ccc} \mathrm{Rep}(G) & \xrightarrow{\Phi_{\mathcal{G}}} & \mathrm{Coh}(\mathcal{O}_U) \\ & \searrow \iota^* & \nearrow \Phi_{\mathcal{H}} \\ & \mathrm{Rep}(H) & \end{array}$$

commutes. Now any reductive subgroup of a reductive group is determined by its invariants in the tensor powers of a faithful self-dual representation [9, prop. 3.1], so if ι were not an isomorphism, then we could in particular find an irreducible representation $W \in \mathrm{Rep}(G)$ whose restriction $\iota^*(W) \simeq W_1 \oplus W_2$ splits into a sum of at least two positive-dimensional pieces. On the geometric side this would mean that we could find a simple holonomic module $\mathcal{N} \in \langle \mathcal{M} \rangle$ such that on U the Fourier-Mukai transform

$$\mathcal{H}^0(\mathrm{FM}(\mathcal{N}))|_U \simeq \mathcal{W}_1 \oplus \mathcal{W}_2$$

splits nontrivially into a direct sum of two vector bundles $\mathcal{W}_1, \mathcal{W}_2 \in \mathrm{Coh}(\mathcal{O}_U)$. By the reconstruction result in [43, cor. 21.3] this splitting would extend to a nontrivial splitting of $\mathrm{FM}(\mathcal{N})$ as a direct sum of two complexes. Since we are working in the algebraic category throughout, we may apply the inverse of the Fourier-Mukai transform to deduce that $\mathcal{N} \simeq \mathcal{N}_1 \oplus \mathcal{N}_2$ splits into a direct sum of two holonomic submodules, contradicting our simplicity assumption. \square

3.c. Almost connectedness. The interpretation of our Tannaka groups in terms of principal bundles easily implies that these groups are almost connected in the sense of theorem 1.3. For perverse sheaves this is a result of Weissauer [46], but our proof is very different from the argument in loc. cit. and also applies to irregular holonomic \mathcal{D}_A -modules. Let us first recall the statement:

Theorem 3.3. *For any $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_A)$ with Tannaka group $G = G(\mathcal{M})$ we have an epimorphism*

$$\pi_1(\hat{A}, 0) \twoheadrightarrow G/G^0$$

from the fundamental group of the dual abelian variety onto the group of connected components of G . In particular G/G^0 is a finite abelian group.

Proof. Let $\pi = G/G^0$ denote the group of connected components and $W = \mathbb{C}[\pi]$ its regular representation. Then W corresponds via the Tannakian formalism to some $\mathcal{N} \in \mathrm{Hol}(\mathcal{D}_A)$ with Tannaka group $G(\mathcal{N}) = \pi$ because π acts faithfully on the regular representation. Replacing the original module \mathcal{M} with \mathcal{N} we may hence assume that the group $G = G(\mathcal{M})$ is itself finite and in particular reductive, so that \mathcal{M} can be assumed to be semisimple.

Now consider as above the algebraic vector bundle $\mathcal{E} = \Phi(\mathcal{M})$ over a Zariski open subset $U \subseteq A^\natural$ whose complement has codimension at least two. By theorem 3.2 this vector bundle has a natural reduction to a principal bundle \mathcal{G} with structure group G . By minimality the total space \mathcal{G} is connected: Otherwise the stabilizer of a connected component $\mathcal{H} \subset \mathcal{G}$ would be a proper subgroup $H \subset G$, and the composite morphism

$$\mathcal{H} \times^H G \hookrightarrow \mathcal{G} \times^H G \twoheadrightarrow \mathcal{G} \times^G G = \mathcal{G}$$

would be an isomorphism of principal bundles. Then \mathcal{G} would admit a reduction to the principal bundle \mathcal{H} with the smaller reductive structure group $H \subset G$, which contradicts the minimality.

As the structure group of our principal bundle is finite, we get that $\mathcal{G} \rightarrow U$ is a connected finite étale Galois cover with Galois group G . Fixing $u \in U(\mathbb{C})$, any such cover is given by an epimorphism

$$\pi_1(U, u) \twoheadrightarrow G = \text{Aut}(\mathcal{G}/U).$$

But $\pi_1(U, u) = \pi_1(A^\natural, u) = \pi_1(\hat{A}, 0)$ since the complement of the subset $U \subseteq A^\natural$ has codimension at least two and since $A^\natural \rightarrow \hat{A}$ is a topologically trivial fibration with contractible fibres, so the result follows. \square

For convenience of the reader we also include the interpretation of direct and inverse images under an isogeny $p : A \rightarrow B$ in terms of restriction and induction functors, which follows from the above with the same arguments as in [46].

Proof of corollary 1.4. The direct and inverse images for p descend to a pair of exact adjoint functors between $M(A)$ and $M(B)$. Since $p^*(\mathcal{N}) = \bigoplus_{a \in \ker(p)} t_a^*(\mathcal{M})$ for the translations $t_a : A \rightarrow A, x \mapsto x + a$, we get a pair

$$p^* : \langle \mathcal{N} \rangle \rightleftarrows \langle p^*(\mathcal{N}) \rangle : p_*$$

of exact adjoint functors. Here the right adjoint p_* is a tensor functor corresponding to the restriction functor for an embedding $\iota : G(\mathcal{N}) \hookrightarrow G(p^*(\mathcal{N}))$, so the left adjoint p^* is the corresponding induction functor. Usually the latter exists only on the level of infinite dimensional algebraic representations. Since here it already exists in the finite dimensional setup, the image of ι is a subgroup of finite index; but any object of $\langle \mathcal{N} \rangle$ is a subquotient of the direct image of some object in $\langle \mathcal{M} \rangle$, so the composite homomorphism $G(\mathcal{N}) \hookrightarrow G(p^*(\mathcal{N})) \rightarrow G(\mathcal{M})$ is injective and the claim follows. \square

4. TENSOR CATEGORIES OF GERMS

We now pass to the proof of our main theorem 1.5. In the next section we will construct an exact tensor functor on $M(A)$ with values in germs of local systems on characteristic varieties. In this preliminary section we introduce possible target categories and relate them to multiplicative subgroups and their normalizers.

4.a. Germs of local systems. The target categories for our tensor functors will be endowed with a convolution product defined via the diagram

$$\begin{array}{ccccc}
 & A^2 \times V & & & \\
 & \swarrow id \times \delta & \parallel & \searrow a \times id & \\
 A^2 \times V^2 & & A^2 \times_A T^*A & & A \times V \\
 \parallel & \swarrow \rho & & \searrow \varpi & \parallel \\
 T^*A^2 & & & & T^*A
 \end{array}$$

where ρ and ϖ are induced by the addition morphism $a : A^2 \rightarrow A$ and by its differential, the diagonal map $\delta : V \hookrightarrow V^2$ on cotangent spaces. Since for the moment we will not use the symplectic structure on the cotangent bundle, we replace $V = H^0(A, \Omega_A^1)$ by any irreducible algebraic variety U and we still denote by

$$\varpi = a \times \text{id} : A^2 \times U \longrightarrow A \times U \quad \text{and} \quad \rho = \text{id} \times \delta : A^2 \times U \longrightarrow A^2 \times U^2$$

the maps induced by the addition morphism and by the diagonal. We are mainly interested in the case where $U \subseteq V$ is an open subset of the cotangent space, but more general situations also arise naturally.

We want to consider local systems on closed subvarieties $\Lambda \subset A \times U$ such that the Gauss map $\gamma : \Lambda \rightarrow U$ given by the projection to the second factor is a finite étale cover. It is natural to define the *convolution* of such subvarieties $\Lambda_1, \Lambda_2 \subset A \times U$ by

$$\Lambda_1 * \Lambda_2 = \varpi(\Lambda_1 \times_U \Lambda_2) \subset A \times U$$

but usually this convolution has singularities and its Gauss map will no longer be a finite étale cover. Nevertheless, locally on any sufficiently small subset $U_0 \subseteq U$ in the classical topology, $\Lambda_1 * \Lambda_2$ is a union of irreducible analytic subvarieties whose Gauss maps are local isomorphisms. So consider the category $\text{LS}(A, U)$ of all sheaves \mathcal{F} of complex vector spaces on $A \times U$ that restrict over any sufficiently small classical open $U_0 \subseteq U$ to a finite sum

$$\mathcal{F}|_{A \times U_0} \simeq \bigoplus_{\alpha} i_{\alpha,*}(\mathcal{F}_{\alpha}),$$

where \mathcal{F}_{α} are finite rank local systems on analytic subvarieties $i_{\alpha} : \Lambda_{\alpha} \hookrightarrow A \times U_0$ that project isomorphically onto U_0 . This category contains all the local systems that we are interested in, and by construction it is stable under the convolution product

$$* : \text{LS}(A, U) \times \text{LS}(A, U) \longrightarrow \text{LS}(A, U), \quad \mathcal{F}_1 * \mathcal{F}_2 = \varpi_* \rho^{-1}(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

For any point $u \in U$ we introduce a category $\text{LS}(A, u)$ of *germs of local systems* as follows: The objects of this category are the pairs $\alpha = (\mathcal{F}_{\alpha}, U_{\alpha})$ consisting of a Zariski open neighborhood $U_{\alpha} \subseteq U$ of u together with an object $\mathcal{F}_{\alpha} \in \text{LS}(A, U_{\alpha})$, and morphisms are defined by

$$\text{Hom}_{\text{LS}(A, u)}(\alpha, \beta) = \varinjlim \text{Hom}_{\text{LS}(A, W)}(\mathcal{F}_{\alpha}|_{A \times W}, \mathcal{F}_{\beta}|_{A \times W})$$

where the limit runs over all Zariski open neighborhoods $W \subseteq U_{\alpha} \cap U_{\beta}$ of u . This is an abelian category, and we equip it with the convolution product induced by the previous one in the obvious way.

Lemma 4.1. *The convolution product naturally endows $\text{LS}(A, U)$ and $\text{LS}(A, u)$ for $u \in U$ with the structure of a rigid abelian tensor category such that the passage to germs is an exact tensor functor $\text{LS}(A, U) \rightarrow \text{LS}(A, u)$, $\mathcal{F} \mapsto (\mathcal{F}, U)$.*

Proof. The convolution product is an exact bifunctor because ϖ is finite on the occurring supports. The unit object for the convolution product on $\text{LS}(A, U)$ is the constant sheaf $\mathbf{1} = \mathbb{C}_{\Lambda_0}$ on $\Lambda_0 = \{0\} \times U$ and $\mathbf{1} \xrightarrow{\sim} \mathbf{1} * \mathbf{1}$ is the obvious unit

isomorphism. We define the commutativity constraint $\psi_{\mathcal{F}_1, \mathcal{F}_2}$ for $\mathcal{F}_i \in \text{LS}(A, U)$ by the composite of the natural isomorphisms

$$\begin{array}{ccc} \mathcal{F}_1 * \mathcal{F}_2 & \xlongequal{\quad} & \varpi_* \rho^{-1}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \longrightarrow \varpi_* \rho^{-1}(\sigma_{A \times U})_*(\mathcal{F}_2 \boxtimes \mathcal{F}_1) \\ \downarrow \psi_{\mathcal{F}_1, \mathcal{F}_2} & & \downarrow \\ \mathcal{F}_2 * \mathcal{F}_1 & \xlongequal{\quad} & \varpi_* \rho^{-1}(\mathcal{F}_2 \boxtimes \mathcal{F}_1) \longleftarrow \varpi_*(\sigma_A \times id_U)_* \rho^{-1}(\mathcal{F}_2 \boxtimes \mathcal{F}_1) \end{array}$$

where for any variety Z we denote by $\sigma_Z : Z \times Z \longrightarrow Z \times Z, (z_1, z_2) \mapsto (z_2, z_1)$ the involution that flips the two factors. The associativity constraint $\varphi_{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3}$ is given by the natural isomorphisms

$$\begin{array}{ccc} (\mathcal{F}_1 * \mathcal{F}_2) * \mathcal{F}_3 & \xlongequal{\quad} & \varpi_* \rho^{-1}((\mathcal{F}_1 * \mathcal{F}_2) \boxtimes \mathcal{F}_3) \longrightarrow \varpi_{3*} \rho_3^{-1}((\mathcal{F}_1 \boxtimes \mathcal{F}_2) \boxtimes \mathcal{F}_3) \\ \downarrow \varphi_{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3} & & \parallel \\ \mathcal{F}_1 * (\mathcal{F}_2 * \mathcal{F}_3) & \xlongequal{\quad} & \varpi_* \rho^{-1}(\mathcal{F}_1 \boxtimes (\mathcal{F}_2 * \mathcal{F}_3)) \longleftarrow \varpi_{3*} \rho_3^{-1}(\mathcal{F}_1 \boxtimes (\mathcal{F}_2 \boxtimes \mathcal{F}_3)) \end{array}$$

where $\rho_3 : A^3 \times U \rightarrow A^3 \times U^3$ and $\varpi_3 : A^3 \times U \rightarrow A \times U$ are the diagonal and the addition morphism. One immediately verifies that with these definitions $\text{LS}(A, U)$ is a rigid abelian tensor category. We endow $\text{LS}(A, u)$ with the tensor structure induced by the previous one so that the passage to germs is a tensor functor. \square

If $\gamma : A \times U \rightarrow U$ denotes the projection, then for any closed point $u \in U(\mathbb{C})$ the functor

$$\text{LS}(A, U) \longrightarrow \text{Vect}(\mathbb{C}), \quad \mathcal{F} \mapsto \gamma_*(\mathcal{F})(u)$$

is a fibre functor which factors over the tensor category of germs $\text{LS}(A, u)$, hence in particular this category is neutral Tannakian. But our definition of $\text{LS}(A, u)$ makes sense for all scheme theoretic points, so we can also work at the generic point $\eta \in U$ instead of choosing a closed point $u \in U(\mathbb{C})$. For finitely generated tensor subcategories one can always specialize from the generic point to a closed point, using that each $\text{LS}(A, u)$ is a full abelian tensor subcategory of $\text{LS}(A, \eta)$:

Lemma 4.2. *Any finitely generated tensor subcategory of $\text{LS}(A, \eta)$ is equivalent to a tensor subcategory of $\text{LS}(A, u)$ for some $u \in U(\mathbb{C})$.*

Proof. Since every object of $\text{LS}(A, \eta)$ is of finite length, any finitely generated tensor subcategory of $\text{LS}(A, \eta)$ has at most countably many isomorphism classes of objects. By the axiom of choice we can replace any small tensor subcategory by one which is *skeletal* [11, rem. 2.8.7], i.e. contains precisely one object in each isomorphism class. So we may assume the given subcategory has only countably many objects. Each of these is defined on some Zariski open dense subset, and we can take $u \in U(\mathbb{C})$ to be any point in the intersection of these subsets. \square

4.b. Germs of vector bundles. For a weaker but more flexible framework one can replace local systems by coherent vector bundles as follows. Let $\text{VB}(A, U)$ be the category of all coherent sheaves $\mathcal{F} \in \text{Coh}(\mathcal{O}_{A \times U})$ with the property that the projection $\gamma : \text{Supp}(\mathcal{F}) \rightarrow U$ is a finite morphism and that $\gamma_*(\mathcal{F})$ is a locally free sheaf. This is not an abelian category, but it is still an *exact* category in the sense that it admits a natural notion of short exact sequences [5] induced by the

embedding $\mathrm{VB}(A, U) \subset \mathrm{Coh}(\mathcal{O}_{A \times U})$, and it is a tensor category with respect to the convolution product

$$\mathcal{F}_1 * \mathcal{F}_2 = \varpi_* \rho^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

For $u \in U$ we obtain an exact category $\mathrm{VB}(A, u)$ of *germs of vector bundles* whose objects are the pairs $\alpha = (\mathcal{F}_\alpha, U_\alpha)$ of a Zariski open neighborhood $U_\alpha \subseteq U$ of u and $\mathcal{F}_\alpha \in \mathrm{VB}(A, U_\alpha)$, with morphisms

$$\mathrm{Hom}_{\mathrm{VB}(A, u)}(\alpha, \beta) = \lim_{\longrightarrow} \mathrm{Hom}_{\mathrm{VB}(A, U)}(\mathcal{F}_\alpha|_{A \times W}, \mathcal{F}_\beta|_{A \times W})$$

where the limit runs over all Zariski open neighborhoods $W \subseteq U_\alpha \cap U_\beta$ of u . We equip these categories of germs with the convolution product induced by the one on $\mathrm{VB}(A, U)$ in the obvious way. Then for $u \in U$ we get a commutative diagram of exact tensor categories

$$\begin{array}{ccc} \mathrm{LS}(A, U) & \longrightarrow & \mathrm{LS}(A, u) \\ \downarrow & & \downarrow \\ \mathrm{VB}(A, U) & \longrightarrow & \mathrm{VB}(A, u) \end{array}$$

in which all arrows are *exact* tensor functors in the sense that they send exact sequences to exact sequences, and except for the bottom one they are faithful.

4.c. Multiplicative subgroups and their normalizers. An algebraic group T is said to be a *multiplicative group* if it embeds into \mathbb{G}_m^r for some $r \in \mathbb{N}$. Then the group $X = \mathrm{Hom}(T, \mathbb{G}_m)$ of algebraic characters is a finitely generated abelian group, and the weight space decomposition gives an equivalence of abelian tensor categories

$$\mathrm{Rep}(T) \simeq \mathrm{Vect}_X(\mathbb{C})$$

where the right hand side denotes the category of finite dimensional X -graded complex vector spaces. The multiplicative group is recovered from its characters as the Cartier dual $T = \mathrm{Hom}(X, \mathbb{G}_m)$, and this sets up an antiequivalence between the categories of multiplicative groups and of finitely generated abelian groups.

So multiplicative subgroups of a linear algebraic group G correspond to fibre functors $\omega : \mathrm{Rep}(G) \rightarrow \mathrm{Vect}_X(\mathbb{C})$, where by a *fibre functor* we mean any faithful exact tensor functor. In looking for such functors we do not require the abelian group X to be finitely generated, but for the definition of ω it can be replaced by the subgroup

$$X(\omega) = \{x \in X \mid \omega(\rho)_x \neq 0 \text{ for some } \rho \in \mathrm{Rep}(G)\}$$

of weights that occur in the considered representations, and this subgroup $X(\omega)$ is finitely generated: It is generated by the set $\{x \in X \mid \omega(\delta)_x \neq 0\}$ for any faithful representation $\delta \in \mathrm{Rep}(G)$ since any faithful representation generates $\mathrm{Rep}(G)$ as a rigid abelian tensor category [10, proof of prop. 2.20b] [9, prop. 3.1a]. We denote by

$$T(\omega) = \mathrm{Hom}(X(\omega), \mathbb{G}_m) \hookrightarrow G$$

the corresponding multiplicative subgroup. If for some linear algebraic group N we have a diagram of tensor functors

$$\begin{array}{ccc} \mathrm{Rep}(G) & \xrightarrow{\omega} & \mathrm{Vect}_X(\mathbb{C}) \\ & \searrow \exists & \nearrow \\ & \mathrm{Rep}_X(N) & \end{array}$$

where $\mathrm{Rep}_X(N)$ denotes the abelian tensor category of algebraic representations of N whose underlying vector space is X -graded such that the action of N permutes the graded pieces, then we get a homomorphism $N \rightarrow N_G(T(\omega))$ to the normalizer of the multiplicative subgroup constructed above.

4.d. Back to Gauss maps. Multiplicative subgroups arise from tensor functors to the categories of germs in section 4.a as follows. For $u \in U(\mathbb{C})$ and α in $\mathrm{VB}(A, u)$ or $\mathrm{LS}(A, u)$, let

$$X(\alpha) = \langle a \in A(\mathbb{C}) \mid (a, u) \in \mathrm{Supp}(\mathcal{F}_\alpha) \rangle \subset A(\mathbb{C})$$

be the subgroup generated by the fibre of the Gauss map $\gamma_\alpha : \mathrm{Supp}(\mathcal{F}_\alpha) \rightarrow U_\alpha$ over u . For germs $\alpha \in \mathrm{LS}(A, u)$ of local systems we also consider the monodromy representation

$$\pi_1(U_\alpha, u) \longrightarrow \mathrm{Gl}(F_\alpha) \quad \text{on the fibre} \quad F_\alpha = \gamma_{\alpha*}(\mathcal{F}_\alpha)(u)$$

and denote the Zariski closure of the image of this representation by $\pi(\alpha)$.

Theorem 4.3. *Let G be a linear algebraic group, and fix a point $u \in U(\mathbb{C})$.*

- (1) *Any exact tensor functor $F : \mathrm{Rep}(G) \rightarrow \mathrm{VB}(A, u)$ gives rise to a closed immersion*

$$T = \mathrm{Hom}(X, \mathbb{G}_m) \hookrightarrow G$$

where $X = X(\alpha)$ for the image $\alpha = F(\delta)$ of any faithful $\delta \in \mathrm{Rep}(G)$.

- (2) *If F factors over an exact tensor functor to $\mathrm{LS}(A, u)$, then we furthermore get an embedding*

$$\pi = \pi(\alpha) \hookrightarrow N_G(T).$$

Proof. Taking the fibre of germs of vector bundles at the point u , we get a tensor functor

$$\omega : \mathrm{VB}(A, u) \longrightarrow \mathrm{Vect}(\mathbb{C}), \quad \alpha = (\mathcal{F}_\alpha, U_\alpha) \mapsto F_\alpha = \gamma_{\alpha*}(\mathcal{F}_\alpha)(u).$$

This is an exact functor because the functor which associates to a coherent vector bundle its fibre at a given point is exact, although it is not faithful. We claim that the tensor functor ω comes with a natural grading by the group $A(\mathbb{C})$ of points on the abelian variety. Indeed, for $\alpha = (\mathcal{F}_\alpha, U_\alpha) \in \mathrm{VB}(A, u)$ the Gauss map γ_α is finite, and

$$(4.4) \quad \omega(\alpha) = \bigoplus_{a \in A(\mathbb{C})} \mathcal{F}_\alpha(a, u)$$

where the sum on the right hand side runs over the finitely many points $a \in A(\mathbb{C})$ with $(a, u) \in \mathrm{Supp}(\mathcal{F}_\alpha)$. One easily checks that this decomposition is compatible

with tensor products; more precisely, for $\alpha, \beta \in \text{VB}(A, u)$ the tensor functoriality gives a natural isomorphism

$$c_{\alpha, \beta} : \omega(\alpha) \otimes \omega(\beta) \xrightarrow{\sim} \omega(\alpha * \beta),$$

and a look at the supports of the relevant sheaves shows that $c_{\alpha, \beta}$ decomposes as a sum of isomorphisms

$$\bigoplus_{a+b=c} \mathcal{F}_\alpha(a, u) \otimes \mathcal{F}_\beta(b, u) \xrightarrow{\sim} \varpi_*(\rho^{-1}(\mathcal{F}_\alpha \boxtimes \mathcal{F}_\beta))(c, u)$$

for $c \in A(\mathbb{C})$. Furthermore, if the tensor category $\text{VB}(A, u)$ is replaced by $\text{LS}(A, u)$, then for any germ $\alpha = (\mathcal{F}_\alpha, U_\alpha) \in \text{LS}(A, u)$ the natural action of the monodromy group $\pi(\alpha)$ on the fibre permutes the summands in (4.4). Let

$$\pi(u) = \varprojlim_{\alpha \in \text{LS}(A, u)} \pi(\alpha)$$

be the proalgebraic group which is the inverse limit of all the above monodromy groups, and let $\text{Rep}_{A(\mathbb{C})}(\pi(u))$ be the category of its finite dimensional algebraic representations which factor over one of the quotients $\pi(u) \rightarrow \pi(\alpha)$ and whose underlying vector space is graded by $A(\mathbb{C})$ such that the graded pieces are permuted by the group action. We get the following diagram of tensor functors:

$$\begin{array}{ccc} \text{LS}(A, u) & \xrightarrow{\omega} & \text{Vect}(\mathbb{C}) \\ \downarrow & \searrow & \uparrow \\ & \text{Rep}_{A(\mathbb{C})}(\pi(u)) & \\ \downarrow & \downarrow & \downarrow \\ \text{VB}(A, u) & \xrightarrow{\quad} & \text{Vect}(\mathbb{C}) \\ & \searrow & \uparrow \\ & \text{Vect}_{A(\mathbb{C})}(\mathbb{C}) & \end{array}$$

Now if G is a linear algebraic group and $F : \text{Rep}(G) \rightarrow \text{VB}(A, u)$ is an exact tensor functor, then the composite

$$\text{Rep}(G) \rightarrow \text{VB}(A, u) \rightarrow \text{Vect}_{A(\mathbb{C})}(\mathbb{C})$$

factors over the category $\text{Vect}_X(\mathbb{C})$, where $X = X(F(\delta))$ for any faithful $\delta \in \text{Rep}(G)$ by section 4.c. Note that the composite is a fibre functor because any exact tensor functor between rigid abelian tensor categories with $\text{End}(\mathbf{1}) = \mathbb{C}$ is automatically faithful [10, prop. 1.19]. This fills in the dashed arrows in the diagram

$$\begin{array}{ccccc} \text{Rep}(G) & \cdots\cdots\cdots & \text{Rep}_X(\pi) & & \\ \parallel & \searrow & \downarrow & \searrow & \\ & \text{LS}(A, u) & \xrightarrow{\quad} & \text{Rep}_{A(\mathbb{C})}(\pi(u)) & \\ & \downarrow & & \downarrow & \\ \text{Rep}(G) & \dashrightarrow & \text{Vect}_X(\mathbb{C}) & & \\ & \searrow & \downarrow & \searrow & \\ & \text{VB}(A, u) & \xrightarrow{\quad} & \text{Vect}_{A(\mathbb{C})}(\mathbb{C}) & \end{array}$$

and thus proves (1). If F factors over an exact tensor functor to $\text{LS}(A, u)$, then (2) follows as well by filling in the dotted arrows for $\pi = \pi(F(\delta))$. \square

4.e. Dependence on the base point. In the above the base point $u \in U(\mathbb{C})$ has been fixed, and it is natural to ask how the multiplicative subgroup in theorem 4.3 depends on it. More generally, for any algebraic subvariety $\Lambda \subset A \times U$ whose Gauss map $\gamma : \Lambda \rightarrow U$ is a finite morphism, we may consider the finitely generated abelian groups

$$X_u = \langle a \in A(\mathbb{C}) \mid (a, u) \in \Lambda \rangle \quad \text{for } u \in U(\mathbb{C}).$$

If Λ is the complement of the zero section in the conormal bundle to a smooth curve of genus two in its Jacobian, then X_u is generated by a point on the curve, hence isomorphic to \mathbb{Z} except for finitely many $u \in U(\mathbb{C})$ [36]. Returning to the general case, we do not know if X_u only jumps on finitely many closed subvarieties, but for trivial reasons it can jump at most on countably many ones:

Lemma 4.5. *The isomorphism type of X_u is constant for very general $u \in U(\mathbb{C})$.*

Proof. Shrinking U we may assume γ is a finite étale cover. Then the number n of points in the fibre $F_u = \{a \in A(\mathbb{C}) \mid (a, u) \in \Lambda\}$ will not depend on u , and locally in the classical topology we may identify all nearby fibres with each other. With these local identifications the subgroup $R_u = \{(c_a)_{a \in F_u} \mid \sum_{a \in F_u} c_a \cdot a = 0\} \subseteq \mathbb{Z}^n$ of relations between points of the fibre is constant for all u outside countably many proper closed subvarieties: For $c = (c_1, \dots, c_n) \in \mathbb{Z}^n$ the corresponding relation defines a closed subvariety

$$S_c = \left\{ (a_i)_{1 \leq i \leq n} \in A^n \mid \sum_{i=1}^n c_i \cdot a_i = 0 \right\} \subseteq A^n = A \times \dots \times A.$$

Let S'_c be the preimage of this subvariety in the n -fold fibered product $\Lambda \times_U \dots \times_U \Lambda$ and denote by $S''_c \subseteq S'_c$ the union of all those irreducible components whose image in U is a proper closed subset. Then the claim of the lemma will hold for all u outside these countably many proper closed subsets. \square

5. MICROLOCALIZATION

We now construct an exact tensor functor from $M(A)$ to the category $LS(A, \eta)$ of germs of local systems. This will prove our main theorem 1.5 by the specialization lemma 4.2 and theorem 4.3. Our tensor functor will arise from microlocal analysis and factors over a certain tensor category $MM(A)$ of microdifferential modules.

5.a. A reminder on microdifferential modules. For any complex manifold X we denote by $\pi : T^*X \rightarrow X$ the projection from the total space of its cotangent bundle. On this total space we have the sheaf \mathcal{E}_X of holomorphic microdifferential operators [39] [17] [41]; recall that this is a sheaf of rings containing $\pi^{-1}(\mathcal{D}_X)$ and that the corresponding categories of right modules are related by the faithful exact functor

$$\text{Mod}(\mathcal{D}_X) \longrightarrow \text{Mod}(\mathcal{E}_X), \quad \mathcal{M} \mapsto \pi^{-1}(\mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{E}_X$$

with

$$\text{Char}(\mathcal{M}) = \text{Supp}(\pi^{-1}(\mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{E}_X).$$

Let $\text{Hol}(\mathcal{E}_X) \subset \text{Mod}(\mathcal{E}_X)$ be the full abelian subcategory of all microdifferential modules which are *holonomic* in the sense that they are coherent and supported on a conic Lagrangian subvariety. We then get a functor $\text{Hol}(\mathcal{D}_X) \rightarrow \text{Hol}(\mathcal{E}_X)$ and similarly a functor $D_{hol}^b(\mathcal{D}_X) \rightarrow D_{hol}^b(\mathcal{E}_X)$ between the corresponding bounded

derived categories. The direct image under a morphism $f : Y \rightarrow X$ is defined to be the functor

$$f_{\dagger} : D_{hol}^b(\mathcal{E}_Y) \longrightarrow D_{hol}^b(\mathcal{E}_X), \quad f_{\dagger}(\mathcal{M}) = R\varpi_*(\rho^{-1}\mathcal{M} \otimes_{\mathcal{E}_Y}^L \mathcal{E}_{Y \rightarrow X}),$$

where for the bimodule $\mathcal{E}_{Y \rightarrow X}$ we refer to [41, sect. I.4.3] and where $\varpi = \varpi_f$ and $\rho = \rho_f$ are the natural morphisms induced by f and by its differential as indicated in the diagram below. Direct images are compatible with composition in the sense that if $g : Z \rightarrow Y$ is a morphism from another complex manifold, then from the diagram

$$\begin{array}{ccccc} & & Z \times_X T^*X & & \\ & \rho_{f,Z} \swarrow & & \searrow \varpi_{g,X} & \\ & Z \times_Y T^*Y & & Y \times_X T^*X & \\ \rho_g \swarrow & & \varpi_g & & \searrow \varpi_f \\ T^*Z & & T^*Y & & T^*X \end{array}$$

we get a natural morphism $\rho_{f,Z}^{-1}(\mathcal{E}_{Z \rightarrow Y}) \otimes_{\mathcal{E}_Y} \varpi_{g,X}^{-1}(\mathcal{E}_{Y \rightarrow X}) \longrightarrow \mathcal{E}_{Z \rightarrow X}$. If f is smooth, then this natural morphism is an isomorphism, and the underived tensor product on the left hand side coincides with the left derived tensor product due to flatness [39, chapt. II, lemma 3.5.1]. If f and g are furthermore proper, we get natural isomorphisms

$$\begin{aligned} f_{\dagger}(g_{\dagger}(\mathcal{M})) &= R\varpi_{f,*}(\rho_f^{-1}(R\varpi_{g,*}(\rho_g^{-1}\mathcal{M} \otimes_{\mathcal{E}_Z}^L \mathcal{E}_{Z \rightarrow Y})) \otimes_{\mathcal{E}_Y}^L \mathcal{E}_{Y \rightarrow X}) \\ &\simeq R\varpi_{fg,*}(\rho_{fg}^{-1}(\mathcal{M}) \otimes_{\mathcal{E}_Z}^L (\rho_{f,Z}^{-1}(\mathcal{E}_{Z \rightarrow Y}) \otimes_{\mathcal{E}_Y} \varpi_{g,X}^{-1}(\mathcal{E}_{Y \rightarrow X}))) \\ &\simeq R\varpi_{fg,*}(\rho_{fg}^{-1}(\mathcal{M}) \otimes_{\mathcal{E}_Z}^L \mathcal{E}_{Z \rightarrow X}) \\ &= (fg)_{\dagger}(\mathcal{M}) \end{aligned}$$

for $\mathcal{M} \in D_{hol}^b(\mathcal{E}_Z)$. The first of the two isomorphisms involves only the projection formula and base change for proper derived direct images of sheaves, and the second one is induced by the natural isomorphism for the transfer bimodules. We are explicit about these isomorphisms since they will be used for the commutativity and associativity constraints in proposition 5.1 below.

5.b. Microlocal convolution. We now specialize to the situation where $X = A$ is an abelian variety, in which case the cotangent bundle T^*A is the trivial bundle with fibre $V = H^0(A, \Omega^1)$. For $\mathcal{M} \in \text{Hol}(\mathcal{E}_A)$ we consider as in section 4.a the Gauss map

$$\gamma : \text{Supp}(\mathcal{M}) \subset A \times V \twoheadrightarrow V$$

which is generically finite of degree $d \in \mathbb{N}_0$ (say). If $d = 0$, then we say that \mathcal{M} is *negligible*. The negligible modules form a Serre subcategory of $\text{Hol}(\mathcal{E}_A)$, and we denote by $\text{MM}(A)$ the abelian quotient category. Similarly, we denote by $\text{DM}(A)$ the Verdier quotient of the triangulated category $D_{hol}^b(\mathcal{E}_A)$ by the thick subcategory of all complexes whose cohomology sheaves are negligible. The universal property of the quotient categories $\text{M}(A)$ and $\text{D}(A)$ in section 1.a gives the dotted arrows in

the following diagram:

$$\begin{array}{ccc}
 & D_{hol}^b(\mathcal{D}_A) & \longrightarrow D(A) \\
 & \nearrow & \downarrow \\
 \text{Hol}(\mathcal{D}_A) & \longrightarrow & M(A) \\
 \downarrow & & \downarrow \\
 & D_{hol}^b(\mathcal{E}_A) & \longrightarrow DM(A) \\
 & \nearrow & \downarrow \\
 \text{Hol}(\mathcal{E}_A) & \longrightarrow & MM(A)
 \end{array}$$

For $\mathcal{M}_1, \mathcal{M}_2 \in D_{hol}^b(\mathcal{E}_A)$ we consider the convolution $\mathcal{M}_1 * \mathcal{M}_2 = a_{\dagger}(M_1 \boxtimes M_2)$.

Proposition 5.1. *The above convolution product endows $D_{hol}^b(\mathcal{E}_A)$ and $DM(A)$ with the structure of a triangulated tensor category in a natural way such that the functors $D_{hol}^b(\mathcal{D}_A) \rightarrow D_{hol}^b(\mathcal{E}_A)$ and $D_{hol}^b(\mathcal{E}_A) \rightarrow DM(A)$ are tensor functors.*

Proof. The main point is to endow the triangulated category $D_{hol}^b(\mathcal{E}_A)$ with a tensor structure with respect to the convolution product. Clearly the convolution product is a \mathbb{C} -linear triangulated bifunctor. If $i : \{0\} \hookrightarrow A$ denotes the inclusion of the origin of the abelian variety, the unit object for the tensor structure will be $\mathbf{1} = i_{\dagger}(\mathbb{C})$. Using the compatibility of direct images with composition, we equip it with the natural isomorphism

$$u : \mathbf{1} = i_{\dagger}(\mathbb{C}) \xrightarrow{\sim} a_{\dagger}(i \times i)_{\dagger}(\mathbb{C} \boxtimes \mathbb{C}) = \mathbf{1} * \mathbf{1}$$

which is induced by the identification $i = a \circ (i \times i)$. To define the commutativity and associativity constraints and to verify the required compatibilities between them, let $\mathcal{M}_1, \dots, \mathcal{M}_4 \in D_{hol}^b(\mathcal{E}_A)$, and for any $n \in \mathbb{N}$, let $a : A^n \rightarrow A$ be the addition morphism. The commutativity constraint $\psi_{\mathcal{M}_1, \mathcal{M}_2}$ is given by the natural isomorphisms in the diagram

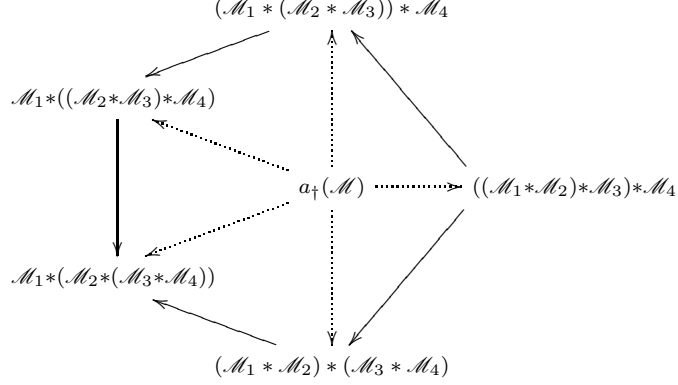
$$\begin{array}{ccc}
 \mathcal{M}_1 * \mathcal{M}_2 & \xrightarrow{\psi_{\mathcal{M}_1, \mathcal{M}_2}} & \mathcal{M}_2 * \mathcal{M}_1 \\
 \parallel & & \parallel \\
 a_{\dagger}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) & \xrightarrow{\sim} a_{\dagger}\sigma_{\dagger}(\mathcal{M}_2 \boxtimes \mathcal{M}_1) \xrightarrow{\sim} & a_{\dagger}(\mathcal{M}_2 \boxtimes \mathcal{M}_1)
 \end{array}$$

where $\sigma : A \times A \rightarrow A \times A, (x, y) \mapsto (y, x)$ is the involution that interchanges the two factors. Similarly, the associativity constraint $\varphi_{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3}$ is given by the natural isomorphisms in the following diagram:

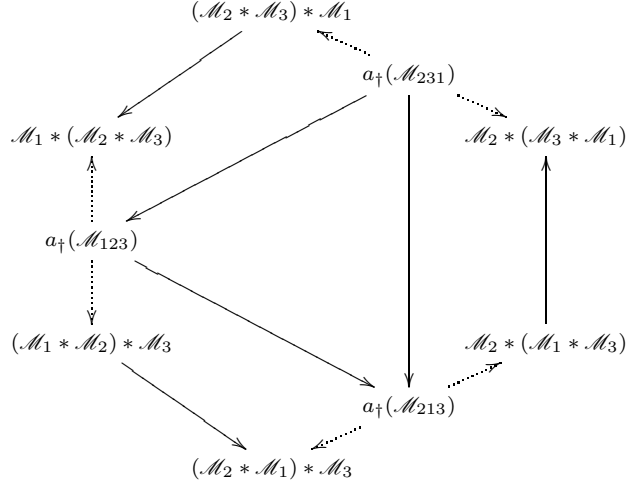
$$\begin{array}{ccc}
 (\mathcal{M}_1 * \mathcal{M}_2) * \mathcal{M}_3 & \xlongequal{\quad} & a_{\dagger}((\mathcal{M}_1 * \mathcal{M}_2) \boxtimes \mathcal{M}_3) \longrightarrow a_{\dagger}((\mathcal{M}_1 \boxtimes \mathcal{M}_2) \boxtimes \mathcal{M}_3) \\
 \downarrow \varphi_{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3} & & \parallel \\
 \mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{M}_3) & \xlongequal{\quad} & a_{\dagger}(\mathcal{M}_1 \boxtimes (\mathcal{M}_2 * \mathcal{M}_3)) \longleftarrow a_{\dagger}(\mathcal{M}_1 \boxtimes (\mathcal{M}_2 \boxtimes \mathcal{M}_3))
 \end{array}$$

It remains to check that these commutativity and associativity constraints satisfy the pentagon and hexagon axioms. Putting $\mathcal{M} = \mathcal{M}_1 \boxtimes \dots \boxtimes \mathcal{M}_4$, the pentagon

axiom boils down to the commutativity of the five triangles in the following diagram of natural isomorphisms:



Similarly, putting $\mathcal{M}_{ijk} = \mathcal{M}_i \boxtimes \mathcal{M}_j \boxtimes \mathcal{M}_k$ for $i, j, k \in \{1, 2, 3\}$, the hexagon axiom boils down to the commutativity of the three trapezoids in the following diagram of natural isomorphisms:



The remark about the natural isomorphism for the composition of direct images at the end of section 5.a then reduces the pentagon and hexagon axioms to obvious compatibility properties for the natural maps between various tensor products of transfer bimodules. Summing up, this shows that $D_{hol}^b(\mathcal{E}_A)$ is a triangulated tensor category with respect to the convolution product. Microlocalization commutes with direct images [42, th. 7.5], so we have natural isomorphisms

$$(\mathcal{N}_1 * \mathcal{N}_2) \otimes_{\mathcal{D}_A} \mathcal{E}_A \xrightarrow{\sim} (\mathcal{N}_1 \otimes_{\mathcal{D}_A} \mathcal{E}_A) * (\mathcal{N}_2 \otimes_{\mathcal{D}_A} \mathcal{E}_A) \quad \text{for } \mathcal{N}_1, \mathcal{N}_2 \in D_{hol}^b(\mathcal{D}_A),$$

and these are compatible with our associativity and commutativity constraints so that they endow the functor $D_{hol}^b(\mathcal{D}_A) \rightarrow D_{hol}^b(\mathcal{E}_A)$ with the structure of a tensor functor. Finally, the convolution product descends to the quotient category $DM(A)$, and we endow the latter with the induced structure of a tensor category so that the quotient functor becomes a tensor functor as well. \square

Corollary 5.2. *The convolution product endows $\mathrm{MM}(A)$ with a natural structure of an abelian tensor category such that the quotient functor $\mathrm{M}(A) \longrightarrow \mathrm{MM}(A)$ is a faithful exact tensor functor.*

Proof. We claim that the essential image of $\mathrm{MM}(A)$ in $\mathrm{DM}(A)$ is stable under the convolution product. For this it suffices to show that for any $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Hol}(\mathcal{E}_A)$, the truncation morphisms

$$\mathcal{M}_1 * \mathcal{M}_2 \longrightarrow \tau_{\geq 0}(\mathcal{M}_1 * \mathcal{M}_2) \longleftarrow \mathcal{H}^0(\mathcal{M}_1 * \mathcal{M}_2)$$

restrict to isomorphisms over some Zariski open dense subset. This holds for any Zariski open dense subset $U \subset V$ over which both $\Lambda_i = \mathrm{Supp}(\mathcal{M}_i)$ are finite since then

$$\varpi : \rho^{-1}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \cap \varpi^{-1}(U) = \Lambda_1 \times_U \Lambda_2 \longrightarrow U$$

is a finite morphism so that [41, th. 3.4.4 and rem. 3.1.7] applies. \square

So if we can find a faithful exact tensor functor $\omega_\eta : \mathrm{MM}(A) \longrightarrow \mathrm{LS}(A, \eta)$ to the category of germs from section 4.a, we will in particular obtain a fibre functor on any finitely generated tensor subcategory of $\mathrm{M}(A)$. For the definition of ω_η we now introduce a reference system of simple holonomic microdifferential modules which behaves well with respect to the convolution product.

5.c. A class of simple test modules. Let $\Omega \subseteq T^*A$ be an open subset. For a sheaf $\mathcal{I} \trianglelefteq \mathcal{E}_A|_\Omega$ of right ideals, one defines the *symbol ideal* $\sigma(\mathcal{I}) \trianglelefteq \mathcal{O}_\Omega$ to be the ideal generated by the principal symbols of all the microdifferential operators in the given ideal. Let us say that \mathcal{I} is *reduced* if its symbol ideal $\sigma(\mathcal{I})$ is the full ideal of functions vanishing on some closed subvariety of Ω . A module $\mathcal{M} \in \mathrm{Mod}(\mathcal{E}_A|_\Omega)$ will be called *simple* if it can be written as a quotient of $\mathcal{E}_A|_\Omega$ by a reduced right ideal. Working on the preimages $\Omega = A \times U$ of suitable open subsets $U \subseteq V$ of the cotangent space, we want to attach to every conic Lagrangian subvariety a simple module in a natural way.

To achieve this, recall that for any closed conic Lagrangian subvariety $\Lambda \subset T^*A$ there exists a Zariski open dense subset $U \subseteq V$ over which Λ is finite étale in the sense that the projection

$$\Lambda|_U = \Lambda \times_V U \subset T^*A|_U = A \times U \twoheadrightarrow U$$

is a finite étale cover (possibly empty). For any such open subset we define a sheaf of right ideals

$$\mathcal{I} = \mathcal{I}_{\Lambda, U} \trianglelefteq \mathcal{E}_{A, U} = \mathcal{E}_A|_\Omega \quad \text{on} \quad \Omega = A \times U$$

as follows. Let us cover $\Omega = A \times U$ by coordinate charts of the form $\Omega_0 = A_0 \times U_0$ where $A_0 \subset A$, $U_0 \subset U$ are open in the classical topology. Choosing the open subsets to be sufficiently small, we may assume that for each coordinate chart Ω_0 one of the following two cases occurs:

- (a) Either $\Omega_0 \cap \Lambda = \emptyset$. In this case we put $\mathcal{I}|_{\Omega_0} = \mathcal{E}_A|_{\Omega_0}$.
- (b) Or $\Omega_0 \cap \Lambda$ projects isomorphically onto U_0 . In this case, let $z = (z_1, \dots, z_g)$ be local coordinates on $A_0 \subset A$ which pull back to affine linear coordinates on the universal cover of the abelian variety A , and define holomorphic functions $f_i : U_0 \rightarrow \mathbb{C}$ by

$$\Omega_0 \cap \Lambda = \{(z, \xi) \in A_0 \times U_0 \mid z_i = f_i(\xi) \text{ for } i = 1, 2, \dots, g\}.$$

Then each f_i is homogenous of degree zero since Λ is conic. So in our chosen coordinate system we may consider $z_i - f_i(\xi)$ as a section of $\mathcal{E}_A(0)|_{\Omega_0}$, and we define

$$\mathcal{J}|_{\Omega_0} = \sum_{i=1}^g (z_i - f_i(\xi)) \cdot \mathcal{E}_A|_{\Omega_0}.$$

The formula for the transformation of microdifferential operators under coordinate changes [18, sect. 7.2], applied to the special case of affine linear coordinate changes where no higher derivatives interfere, shows that the definition in (b) does not depend on the choice of the affine linear coordinates. Hence by patching we obtain a sheaf $\mathcal{J} = \mathcal{J}_{\Lambda,U} \subseteq \mathcal{E}_{A,U}$ of right ideals.

Lemma 5.3. *In the above setting, the symbol ideal $\sigma(\mathcal{J}_{\Lambda,U}) \subseteq \mathcal{O}_{\Omega}$ coincides with the ideal of all functions vanishing on $\Omega \cap \Lambda$. Hence the quotient by it defines a simple module*

$$\mathcal{C}_{\Lambda,U} = \mathcal{J}_{\Lambda,U} \backslash \mathcal{E}_{A,U} \in \text{Hol}(\mathcal{E}_A|_{\Omega}) \quad \text{with} \quad \text{Supp}(\mathcal{C}_{\Lambda,U}) = \Omega \cap \Lambda.$$

Proof. The claim is local, so we may assume that we are in case (b) of the definition where the right ideal $\mathcal{J}_{\Lambda,U}|_{\Omega_0} \subseteq \mathcal{E}_A|_{\Omega_0}$ is generated by the homogenous functions $P_i(z, \xi) = z_i - f_i(\xi)$, considered as sections of $\mathcal{E}_A(0)|_{\Omega_0}$. Since these generators depend only linearly on the variables z_i , the Leibniz rule for the product of microdifferential operators shows that the commutators $[P_i, P_k] \in \mathcal{E}_A(-1)|_{\Omega_0}$ coincide in our linear coordinates with their principal symbol, which is given by the Poisson bracket

$$[P_i, P_k] = \{P_i, P_k\} = \sum_j \frac{\partial P_i}{\partial \xi_j} \frac{\partial P_k}{\partial z_j} - \frac{\partial P_k}{\partial \xi_j} \frac{\partial P_i}{\partial z_j} = \frac{\partial f_i}{\partial \xi_k} - \frac{\partial f_k}{\partial \xi_i}.$$

The function on the right hand side depends only on the variables ξ_j but not on the variables z_j . On the other hand, since $\Lambda_0 = \Omega_0 \cap \Lambda$ is an isotropic subvariety, the ideal of functions vanishing on it is stable under the Poisson bracket. Thus on the right hand side of the above equation we have a function on $\Omega_0 = A \times U_0$ which only depends on U_0 but nevertheless vanishes on Λ_0 . This is possible only for the zero function since by assumption Λ_0 is finite étale over U_0 . Therefore the chosen generators for the right ideal $\mathcal{J}_{\Lambda,U}$ commute with each other and the claim follows from [41, prop. I.4.1.5]. \square

Note that the simple module $\mathcal{C}_{\Lambda,U}$ comes with a canonical section on $\Omega = A \times U$, the class of $1 \in \mathcal{E}_A$. We will use these sections to establish a compatibility property for convolutions of our simple modules. Let $\Lambda_1, \Lambda_2 \subset T^*A = A \times V$ be conic Lagrangian subvarieties, and consider the conic Lagrangian subvariety

$$\Lambda_1 * \Lambda_2 = \varpi(\rho^{-1}(\Lambda_1 \times \Lambda_2)) \subset A \times V$$

as in section 4.a. Let $U \subseteq V$ be any open subset over which Λ_1, Λ_2 and $\Lambda_1 * \Lambda_2$ are finite étale so that the corresponding simple modules exist over this subset.

Lemma 5.4. *We have a natural homomorphism*

$$\iota_{\Lambda_1, \Lambda_2} : \mathcal{C}_{\Lambda_1 * \Lambda_2, U} \longrightarrow \mathcal{C}_{\Lambda_1, U} * \mathcal{C}_{\Lambda_2, U}.$$

Proof. Put $\Lambda = \Lambda_1 * \Lambda_2$. The homomorphism will be defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{E}_A|_{A \times U} & \xrightarrow{P \mapsto u \cdot P} & \mathcal{C}_{\Lambda_1, U} * \mathcal{C}_{\Lambda_2, U} \\ & \searrow P \mapsto 1 \cdot P & \nearrow \exists! \\ & \mathcal{C}_{\Lambda, U} & \end{array}$$

where

$$u = \varpi_*(\rho^{-1}(1 \boxtimes 1) \otimes 1_{A^2 \rightarrow A}) \in H^0(A \times U, \mathcal{C}_{\Lambda_1, U} * \mathcal{C}_{\Lambda_2, U}).$$

To see that such a factorization exists, we must show that the ideal sheaf $\mathcal{I}_{\Lambda, U}$ is contained in the annihilator of u . This is a local question, so it suffices to show it on $\Omega_0 = A \times U_0$ for small open subsets $U_0 \subset U$ in the classical topology. We may assume

$$\Omega_0 \cap \Lambda_1 = \bigsqcup_{\alpha} \Lambda_{1, \alpha} \quad \text{and} \quad \Omega_0 \cap \Lambda_2 = \bigsqcup_{\beta} \Lambda_{2, \beta},$$

where

$$\Lambda_{1, \alpha} = \{(g_{1, \alpha}(\xi), \xi) \mid \xi \in U_0\} \quad \text{and} \quad \Lambda_{2, \beta} = \{(g_{2, \beta}(\xi), \xi) \mid \xi \in U_0\}$$

are the graphs of holomorphic maps $g_{1, \alpha} : U_0 \rightarrow A$ and $g_{2, \beta} : U_0 \rightarrow A$. The decomposition of the supports into disjoint closed subsets gives decompositions as direct sums

$$\begin{aligned} \mathcal{C}_{\Lambda_1, U_0} &= \bigoplus_{\alpha} \mathcal{C}_{1, \alpha} \quad \text{with} \quad \text{Supp}(\mathcal{C}_{1, \alpha}) = \Lambda_{1, \alpha}, \\ \mathcal{C}_{\Lambda_2, U_0} &= \bigoplus_{\beta} \mathcal{C}_{2, \beta} \quad \text{with} \quad \text{Supp}(\mathcal{C}_{2, \beta}) = \Lambda_{2, \beta}. \end{aligned}$$

Let $1_{\alpha} \in \mathcal{C}_{1, \alpha}$, $1_{\beta} \in \mathcal{C}_{2, \beta}$ be the components of the sections $1 \in \mathcal{C}_{\Lambda_1, U_0}$, $1 \in \mathcal{C}_{\Lambda_2, U_0}$ in the respective direct summands. Then it will be enough to show that the right ideal $\mathcal{I}_{\Lambda, U_0}$ kills the section

$$u_{\alpha, \beta} = \varpi_*(\rho^{-1}(1_{\alpha} \boxtimes 1_{\beta}) \otimes 1_{A^2 \rightarrow A}) \in \mathcal{C}_{1, \alpha} * \mathcal{C}_{2, \beta}$$

for all α, β . To this end, note that locally near any point of $\varpi(\rho^{-1}(\Lambda_{1, \alpha} \times \Lambda_{2, \beta}))$, we can write Λ as the graph of $f = g_{1, \alpha} + g_{2, \beta} : U_0 \rightarrow A$. Hence locally near any such point we have

$$\mathcal{I}_{\Lambda, U_0} = \sum_{i=1}^g (z_i - f_i(\xi)) \cdot \mathcal{E}_A|_{\Omega_0}$$

where z_i is the i -th coordinate in a local coordinate system (z, ξ) on $T^*A = A \times V$ that arises from a suitable affine linear coordinate system on the universal cover and where f_i denotes the i -th component of f . For a suitable local affine linear coordinate system (z_1, z_2, ξ_1, ξ_2) on the product $T^*A^2 = A^2 \times V^2$, we have in $\mathcal{E}_{A^2 \rightarrow A}$ the relations

$$\begin{aligned} 1_{A^2 \rightarrow A} \cdot z_i &= (z_{1, i} + z_{2, i}) \cdot 1_{A^2 \rightarrow A}, \\ 1_{A^2 \rightarrow A} \cdot h(\xi) &= h(\xi_1) \cdot 1_{A^2 \rightarrow A} = h(\xi_2) \cdot 1_{A^2 \rightarrow A} \end{aligned}$$

for any homogenous holomorphic function $h(\xi)$, considered as a microdifferential operator. Thus

$$1_{A^2 \rightarrow A} \cdot (z_i - f_i(\xi)) = (z_{1, i} - g_{1, \alpha, i}(\xi_1) + z_{2, i} - g_{2, \beta, i}(\xi_2)) \cdot 1_{A^2 \rightarrow A},$$

and it follows that $u_{\alpha, \beta} \cdot (z_i - f_i(\xi)) = 0$ as required. \square

5.d. Second microlocalization. We now pass from holonomic microdifferential modules to local systems on the smooth locus of their support. This is sometimes called the second microlocalization in the literature, but we need an untwisted version of it to make it compatible with the convolution product. To define the required tensor functor we apply a change of coefficients $\mathcal{M} \mapsto \mathcal{M}^{\mathbb{R}} = \mathcal{M} \otimes_{\mathcal{E}_A} \mathcal{E}_A^{\mathbb{R}}$ via the ring extension $\mathcal{E}_A \subset \mathcal{E}_A^{\mathbb{R}}$. For the definition of this ring extension we refer to [17, sect. 1.4] and only quote theorem 3.2.1 of loc. cit.:

*Let $\Omega \subset T^*A$ be an open subset and $\mathcal{C}_\Lambda \in \text{Mod}(\mathcal{E}_A|_\Omega)$ a simple microdifferential module whose support is a smooth Lagrangian subvariety $\Lambda \subset \Omega$. Then for any coherent module $\mathcal{M} \in \text{Mod}(\mathcal{E}_A|_\Omega)$ supported on Λ we have $\mathcal{E}xt_{\mathcal{E}_A}^i(\mathcal{C}_\Lambda, \mathcal{M}^{\mathbb{R}}) = 0$ for all $i \neq 0$, and*

$$\mathcal{H}om_{\mathcal{E}_A}(\mathcal{C}_\Lambda, \mathcal{M}^{\mathbb{R}})$$

is a local system on Λ whose rank is equal to the multiplicity of \mathcal{M} along Λ .

We will apply this result for the simple modules $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda, U}$ from the previous section, omitting the open subset $U \subseteq V$ from the notation when there is no risk of confusion. By the same abuse of notation we will also omit the open subset for objects of the category $\text{LS}(A, \eta)$ of germs from section 4.a. We are interested in the functor

$$F : \text{MM}(A) \longrightarrow \text{LS}(A, \eta), \quad \mathcal{M} \mapsto \mathcal{H}om_{\mathcal{E}_A}(\mathcal{C}_\Lambda, \mathcal{M}^{\mathbb{R}})|_U$$

where on the right hand side $U \subseteq V$ is taken to be the maximal Zariski open subset on which $\Lambda = \text{Supp}(\mathcal{M})$ is finite étale. The functor F is faithful and exact, and our choice of the simple modules in the previous section implies:

Theorem 5.5. *The functor $F : \text{MM}(A) \longrightarrow \text{LS}(A, \eta)$ underlies a tensor functor.*

Proof. We must find isomorphisms

$$c_{\mathcal{M}_1, \mathcal{M}_2} : F(\mathcal{M}_1) * F(\mathcal{M}_2) \xrightarrow{\sim} F(\mathcal{M}_1 * \mathcal{M}_2) \quad \text{for } \mathcal{M}_1, \mathcal{M}_2 \in \text{MM}(A)$$

which are functorial and compatible with the associativity, commutativity and unit constraints. Put $\Lambda_i = \text{Supp}(\mathcal{M}_i)$, and let $U \subset V$ be the maximal open subset where both \mathcal{M}_i are defined and over which the Gauss maps for Λ_1, Λ_2 and $\Lambda = \Lambda_1 * \Lambda_2$ are finite étale. Writing

$$\mathcal{M}_{12}^{\mathbb{R}} = \mathcal{M}_1^{\mathbb{R}} \boxtimes \mathcal{M}_2^{\mathbb{R}} \quad \text{and} \quad \Lambda_{12} = \Lambda_1 \times \Lambda_2,$$

we define $c_{\mathcal{M}_1, \mathcal{M}_2}$ by the following diagram. Here ① is induced by the Künneth morphism

$$\mathcal{H}om_{\mathcal{E}_A}(\mathcal{C}_{\Lambda_1}, \mathcal{M}_1^{\mathbb{R}}) \boxtimes \mathcal{H}om_{\mathcal{E}_A}(\mathcal{C}_{\Lambda_2}, \mathcal{M}_2^{\mathbb{R}}) \longrightarrow \mathcal{H}om_{\mathcal{E}_{A \times A}}(\mathcal{C}_{\Lambda_1} \boxtimes \mathcal{C}_{\Lambda_2}, \mathcal{M}_1^{\mathbb{R}} \boxtimes \mathcal{M}_2^{\mathbb{R}}),$$

while

- ② is the natural restriction map for ρ ,
- ③ is the tensor product with the inclusion $\mathcal{E}_{A^2 \rightarrow A} \hookrightarrow \mathcal{E}_{A^2 \rightarrow A}^{\mathbb{R}}$,
- ④ is induced by the map $\varpi^{-1}(\mathcal{C}_\Lambda) \rightarrow \rho^{-1}(\mathcal{C}_{\Lambda_{12}}) \otimes \mathcal{E}_{A^2 \rightarrow A}$ from lemma 5.4,
- ⑤ is the adjunction isomorphism,
- ⑥ is given by the identity $(\mathcal{M}_1 \boxtimes \mathcal{M}_2)^{\mathbb{R}} = \mathcal{M}_1^{\mathbb{R}} \boxtimes \mathcal{M}_2^{\mathbb{R}}$ and by the compatibility of the functor $(-)^{\mathbb{R}}$ with direct images on the locus where the morphism ϖ is finite [39, chapt. II, th. 3.5.5].

$$\begin{array}{ccc}
F(\mathcal{M}_1) * F(\mathcal{M}_2) & \xrightarrow[\textcircled{1}]{\sim} & \varpi_* \rho^{-1} \mathcal{H}om_{\mathcal{E}_{A^2}}(\mathcal{C}_{\Lambda_1} \boxtimes \mathcal{C}_{\Lambda_2}, \mathcal{M}_1^{\mathbb{R}} \boxtimes \mathcal{M}_2^{\mathbb{R}})|_U \\
\vdots & & \downarrow \textcircled{2} \\
& & \varpi_* \mathcal{H}om_{\rho^{-1} \mathcal{E}_{A^2}}(\rho^{-1} \mathcal{C}_{\Lambda_{12}}, \rho^{-1} \mathcal{M}_{12}^{\mathbb{R}})|_U \\
& & \downarrow \textcircled{3} \\
& & \varpi_* \mathcal{H}om_{\varpi^{-1} \mathcal{E}_A}(\rho^{-1} \mathcal{C}_{\Lambda_{12}} \otimes \mathcal{E}_{A^2 \rightarrow A}, \rho^{-1} \mathcal{M}_{12}^{\mathbb{R}} \otimes \mathcal{E}_{A^2 \rightarrow A}^{\mathbb{R}})|_U \\
& & \downarrow \textcircled{4} \\
& & \varpi_* \mathcal{H}om_{\varpi^{-1} \mathcal{E}_A}(\varpi^{-1} \mathcal{C}_{\Lambda}, \rho^{-1} \mathcal{M}_{12}^{\mathbb{R}} \otimes \mathcal{E}_{A^2 \rightarrow A}^{\mathbb{R}})|_U \\
& & \downarrow \textcircled{5} \\
F(\mathcal{M}_1 * \mathcal{M}_2) & \xrightarrow[\textcircled{6}]{\sim} & \mathcal{H}om_{\mathcal{E}_A}(\mathcal{C}_{\Lambda}, \varpi_*(\rho^{-1} \mathcal{M}_{12}^{\mathbb{R}} \otimes \mathcal{E}_{A^2 \rightarrow A}^{\mathbb{R}}))|_U
\end{array}$$

$c_{\mathcal{M}_1, \mathcal{M}_2}$

Note that $c_{\mathcal{M}_1, \mathcal{M}_2}$ is a morphism between local systems of the same rank. To see that it is an isomorphism, it suffices to show that the maps ① – ⑤ are all injective. This can be checked locally in the classical topology, so in what follows we will work on a classical open subset of the form $\Omega_0 = A \times U_0$ as in the proof of lemma 5.4. Here U_0 can be arbitrarily small, so by the local classification of holonomic microdifferential modules in [17, th. 3.2.1] we may assume each $\mathcal{M}_i^{\mathbb{R}}|_{\Omega_0}$ is isomorphic to a finite direct sum of copies of the module $\mathcal{C}_{\Lambda_i}^{\mathbb{R}}|_{\Omega_0}$. Since ① – ⑤ are functorial under $\mathcal{E}_A^{\mathbb{R}}$ -module homomorphisms and compatible with direct sums, it only remains to show their injectivity in the special case $\mathcal{M}_i = \mathcal{C}_{\Lambda_i}$. In this special case we can omit the superscript $(-)^{\mathbb{R}}$ everywhere without changing the occurring $\mathcal{H}om$ -sheaves. The Künneth morphism inducing ① is then obviously an isomorphism, while the composite of ② – ⑤ sends the identity $id : \mathcal{C}_{\Lambda_{12}} \rightarrow \mathcal{C}_{\Lambda_{12}}$ to the morphism $\mathcal{C}_{\Lambda} \rightarrow \mathcal{C}_{\Lambda_1} * \mathcal{C}_{\Lambda_2}$ from lemma 5.4. One easily sees that this composite is injective and hence an isomorphism.

It remains to check that the $c_{\mathcal{M}_1, \mathcal{M}_2}$ satisfy the usual compatibilities with the unit, commutativity and associativity constraints of $\text{MM}(A)$ and $\text{LS}(A, \eta)$. Since all the morphisms in the relevant diagrams are defined globally, the commutativity of the diagrams can be checked locally in the classical topology. For any section f of $F(\mathcal{M}_1) * F(\mathcal{M}_2)$ over a classical open neighborhood $U_0 \subset V$ of a given point, we may after further shrinking the neighborhood assume that $f = \varpi_* \rho^{-1}(f_1 \boxtimes f_2)$ with $f_i : \mathcal{C}_{\Lambda_i} \rightarrow \mathcal{M}_i^{\mathbb{R}}$ locally defined over U_0 . But in this case the definitions imply that the local section $c_{\mathcal{M}_1, \mathcal{M}_2}(f)$ of $F(\mathcal{M}_1 * \mathcal{M}_2) = \mathcal{H}om_{\mathcal{E}_A}(\mathcal{C}_{\Lambda}, (\mathcal{M}_1 * \mathcal{M}_2)^{\mathbb{R}})|_{U_0}$ is the composite map

$$c_{\mathcal{M}_1, \mathcal{M}_2}(f) : \mathcal{C}_{\Lambda} \xrightarrow{\iota_{\Lambda_1, \Lambda_2}} \mathcal{C}_{\Lambda_1} * \mathcal{C}_{\Lambda_2} \xrightarrow{f_1 * f_2} \mathcal{M}_1^{\mathbb{R}} * \mathcal{M}_2^{\mathbb{R}} \xrightarrow{\sim} (\mathcal{M}_1 * \mathcal{M}_2)^{\mathbb{R}}$$

The compatibility with the unit, commutativity and associativity constraints then boils down to simple properties of the maps ι from lemma 5.4. For instance, the

compatibility with the associativity constraints means that for any conic Lagrangian subvarieties $\Lambda_1, \Lambda_2, \Lambda_3 \subset A \times V$ that are finite étale over a Zariski open $U \subset V$ which is small enough so that the convolutions $\Lambda_i * \Lambda_j$ and $\Lambda_i * \Lambda_j * \Lambda_k$ are still finite étale over it, the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{C}_{\Lambda_1 * \Lambda_2 * \Lambda_3, U} & \xrightarrow{\iota_{\Lambda_1, \Lambda_2 * \Lambda_3}} & \mathcal{C}_{\Lambda_1, U} * \mathcal{C}_{\Lambda_2 * \Lambda_3, U} & \xrightarrow{id * \iota_{\Lambda_2, \Lambda_3}} & \mathcal{C}_{\Lambda_1, U} * (\mathcal{C}_{\Lambda_2, U} * \mathcal{C}_{\Lambda_3, U}) \\
\parallel & & & & \uparrow \varphi \\
\mathcal{C}_{\Lambda_1 * \Lambda_2 * \Lambda_3, U} & \xrightarrow{\iota_{\Lambda_1 * \Lambda_2, \Lambda_3}} & \mathcal{C}_{\Lambda_1 * \Lambda_2, U} * \mathcal{C}_{\Lambda_3, U} & \xrightarrow{\iota_{\Lambda_1, \Lambda_2} * id} & (\mathcal{C}_{\Lambda_1, U} * \mathcal{C}_{\Lambda_2, U}) * \mathcal{C}_{\Lambda_3, U}
\end{array}$$

But this follows easily by looking at the images of the unit section. \square

Remark 5.6. Even if $\Lambda \subset T^*A$ is smooth, the simple test modules $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda, U}$ have only been defined over the open subset $U \subseteq V$ where the Gauss map $\gamma : \Lambda \rightarrow V$ restricts to a finite étale cover. There is an intrinsic reason why in general they cannot be extended as simple modules over the branch locus of this map: If such an extension of \mathcal{C}_Λ exists, then for any module $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ with $\text{CC}(\mathcal{M}) = \Lambda$ the germ

$$F(\pi^{-1}(\mathcal{M}) \otimes_{\mathcal{D}_A} \mathcal{C}_A) \in \text{LS}(A, \eta)$$

extends to a local system \mathcal{F} on the complement of the zero section in Λ , hence its monodromy along small loops $\alpha : [0, 1] \rightarrow \Lambda$ around any branch point $p \in \Lambda$ of the Gauss map must be trivial. Suppose for example that γ is a nontrivial cover of degree $\deg(\gamma) = 2$. Then the triviality of the local monodromy near p would imply that in a suitable basis the local monodromy of $\gamma_*(\mathcal{F})$ near the image point $\gamma(p)$ contains the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which interchanges the stalks at the two points of the fibre $\gamma^{-1}(\gamma(p))$. Since F is a tensor functor, it would follow that this permutation matrix lies in the Tannaka group $G(\mathcal{M})$. But up to a translation we can always assume $G(\mathcal{M}) \subseteq \text{Sl}_2(\mathbb{C})$.

6. APPENDIX: A TWISTOR VARIANT

In this appendix we sketch a twistor variant of the Fourier-Mukai transform that gives another construction of multiplicative subgroups of Tannaka groups. We do not know how to see Weyl groups in this twistor approach, but for \mathcal{D}_A -modules that are not of geometric origin, it usually yields bigger multiplicative subgroups than the above microlocal approach (see example 6.5).

6.a. Twistor modules. The framework of pure twistor modules [38] [31] [32] gives a natural way to pass from semisimple holonomic \mathcal{D}_X -modules on a smooth projective variety X to coherent sheaves on the cotangent bundle T^*X . While the resulting coherent sheaves are algebraic, the constructions are analytic and we will indicate analytic objects by a superscript *an*. Let \mathbb{A}^1 be the affine line with coordinate z . Inside the sheaf of holomorphic relative differential operators we have the subalgebra

$$\mathcal{R}_X^{\text{an}} = \langle \mathcal{O}_{X \times \mathbb{A}^1}^{\text{an}} \oplus z \cdot \mathcal{I}_{X \times \mathbb{A}^1 / \mathbb{A}^1}^{\text{an}} \rangle \subset \mathcal{D}_{X \times \mathbb{A}^1 / \mathbb{A}^1}^{\text{an}}$$

generated by the holomorphic functions and by the relative vector fields vanishing at $z = 0$. This *analytic Rees algebra* is a deformation from holomorphic differential operators to holomorphic fibrewise polynomial functions on the cotangent bundle in the sense that

$$\mathcal{R}_X^{an}/(z-1) \simeq \mathcal{D}_X^{an} \quad \text{and} \quad \mathcal{R}_X^{an}/(z) \simeq \text{Sym}^\bullet(\mathcal{T}_X^{an}).$$

Note that since we are working on a projective variety, the categories of holonomic respectively coherent modules for these two sheaves of rings are equivalent to their algebraic cousins

$$\text{Hol}(\mathcal{D}_X^{an}) \simeq \text{Hol}(\mathcal{D}_X) \quad \text{and} \quad \text{Coh}(\text{Sym}^\bullet(\mathcal{T}_X^{an})) \simeq \text{Coh}(\mathcal{O}_{T^*X}).$$

A first interpolation between these categories is the abelian category $\text{Hol}(\mathcal{R}_X^{an})$ of holonomic right \mathcal{R}_X^{an} -modules [38, def. 1.2.4], but it is too large to have good functorial properties. Motivated by the framework of Hodge modules, Sabbah and Mochizuki introduced a much better interpolation: The abelian category $\text{MT}(X, w)$ of wild polarizable pure *twistor modules* of weight $w \in \mathbb{Z}$. We refer to chapter 4 in loc. cit. for the precise definitions and only recall that we have a faithful exact functor $\text{MT}(X, w) \rightarrow \text{Hol}(\mathcal{R}_X^{an})$, denoted $\mathcal{M} \mapsto \mathcal{M}''$ in what follows, that gives a diagram

$$\begin{array}{ccc} & \xrightarrow{\Xi_{\text{DR}}} & \text{Hol}(\mathcal{D}_X) \\ \text{MT}(X, w) & \longrightarrow & \text{Hol}(\mathcal{R}_X^{an}) \begin{array}{l} \nearrow i_1^* \\ \searrow i_0^* \end{array} \\ & \xrightarrow{\Xi_{\text{Dol}}} & \text{Coh}(\mathcal{O}_{T^*X}) \end{array} \quad \text{over} \quad \begin{array}{c} \{1\} \\ \downarrow i_1 \\ \mathbb{A}^1 \\ \uparrow i_0 \\ \{0\} \end{array}$$

Here $\Xi_{\text{DR}} : \text{MT}(X, w) \xrightarrow{\sim} \text{Hol}(\mathcal{D}_X)^{ss} \subset \text{Hol}(\mathcal{D}_X)$ is an equivalence between the abelian category of wild polarizable pure twistor modules of weight w and the one of semisimple holonomic \mathcal{D}_X -modules [32, th. 19.4.1].

6.b. Some functorial properties. Even though we are ultimately interested in the pure case, it seems convenient to discuss functorial properties in the larger framework of the abelian categories $\text{MTM}(X)$ of algebraic *mixed twistor modules* from [33, sect. 13]. Inside the bounded derived category of the category of mixed twistor modules we consider the full subcategory $\text{D}_X \subset \text{D}^b(\text{MTM}(X))$ of all finite sums $\bigoplus_{w \in \mathbb{Z}} M_w[-w]$ with $M_w \in \text{MT}(X, w)$; the notational clash with section 1.a is deliberate because the constructions in that section also work in the present twistor setting. The category D_X contains all the pure twistor modules of weight zero and comes with the functors we want: We have an external product

$$\boxtimes : \text{D}_X \times \text{D}_X \longrightarrow \text{D}_{X \times X}$$

that is compatible with the usual external product via the functors Ξ_{DR} and Ξ_{Dol} , and any morphism $f : Y \rightarrow X$ of smooth projective varieties induces a direct image functor

$$f_{\dagger} : \text{D}_Y \longrightarrow \text{D}_X$$

that is compatible with the usual direct image functor for \mathcal{R} - and \mathcal{D} -modules. For the behaviour of Ξ_{Dol} under direct images we have the following twistor analog of Laumon's formula from [35, th. 2.4] [28, construction 2.3.2].

Proposition 6.1. *Let $\varpi = \varpi_f$ and $\rho = \rho_f$ be the morphisms induced by f and by its differential as in section 5.a. Then for any $\mathcal{M} \in \text{D}_Y$ we have a natural isomorphism*

$$\Xi_{\text{Dol}}(f_+(\mathcal{M})) \xrightarrow{\sim} R\varpi_* L\rho^*(\Xi_{\text{Dol}}(\mathcal{M})) \quad \text{in} \quad \text{D}_{\text{coh}}^b(\mathcal{O}_{T^*X}).$$

Proof. Passing to direct summands we may assume that $\mathcal{M} \in \text{MT}(Y, w)$ is a single twistor module. Let $\mathcal{M}'' \in \text{Hol}(\mathcal{R}_Y^{\text{an}})$ be the underlying Rees module. The graph factorization reduces us to the case where f is either a closed embedding or a projection, and closed embeddings can be treated directly. So let us consider a projection $f : Y = X \times Z \rightarrow X$. By [38, ex. 1.4.1] the relative Spencer complex gives

$$f_+(\mathcal{M}'') \simeq Rf_*(\mathcal{M}'' \otimes_{\mathcal{O}_{Y \times \mathbb{A}^1}}^{\text{an}} \wedge^{-\bullet}(z \cdot \mathcal{T}_{Y \times \mathbb{A}^1/X \times \mathbb{A}^1}^{\text{an}})),$$

where the derived direct image functor on the right hand side can be computed via a Dolbeault resolution as in loc. cit. So far this holds for any $\mathcal{M}'' \in \text{Hol}(\mathcal{R}_Y^{\text{an}})$. In our situation \mathcal{M}'' underlies a polarizable pure twistor module, so by [32, th. 18.1.1] all the cohomology sheaves $\mathcal{H}^i(f_+(\mathcal{M}''))$ underlie polarizable pure twistor modules as well. Then in particular all these cohomology sheaves are *strict* in the sense that the multiplication by z is injective on them. The individual terms of the Dolbeault resolution which computes the direct image are also strict. So lemma 6.2 gives a natural isomorphism

$$Li_0^*(f_+(\mathcal{M}'')) \simeq Rf_*(i_0^*(\mathcal{M}'') \otimes_{\mathcal{O}_Y^{\text{an}}} \wedge^{-\bullet}(\mathcal{T}_{Y/X}^{\text{an}}))$$

in $\text{D}_{\text{coh}}^b(\text{Sym}^\bullet(\mathcal{T}_X^{\text{an}}))$. The rest of the proof works in the same way as in [35, th. 2.4] with the appropriate translation from left to right modules. \square

For completeness we include the following remark that has been used above, putting $\mathcal{R} = \mathcal{R}_X^{\text{an}}$ for brevity:

Lemma 6.2. *Let $K = [\cdots \rightarrow K^\nu \rightarrow K^{\nu+1} \rightarrow \cdots]$ be a bounded complex of strict modules in $\text{Mod}(\mathcal{R})$, and assume that the cohomology sheaves $\mathcal{H}^\nu(K)$ are strict for all $\nu \in \mathbb{Z}$. Then in the derived category $\text{D}^b(\text{Mod}(\mathcal{R}/(z)))$ we have a natural isomorphism*

$$K \otimes_{\mathcal{R}}^L \mathcal{R}/(z) \simeq [\cdots \rightarrow K^\nu/zK^\nu \rightarrow K^{\nu+1}/zK^{\nu+1} \rightarrow \cdots].$$

Proof. Let us denote by K/zK the complex on the right hand side. The quotient map $q : K \rightarrow K/zK$ induces on cohomology sheaves a map that factors uniquely over maps r_ν as indicated in the following diagram:

$$\begin{array}{ccc} \mathcal{H}^\nu(K) & \xrightarrow{\mathcal{H}^\nu(q)} & \mathcal{H}^\nu(K/zK) \\ & \searrow & \nearrow \exists! r_\nu \\ & \mathcal{H}^\nu(K) \otimes_{\mathcal{R}} \mathcal{R}/(z) & \end{array}$$

A simple diagram chase, using that the cohomology sheaf $\mathcal{H}^{\nu+1}(K)$ as well as the terms K^ν and $K^{\nu+1}$ are strict, shows that r_ν is an isomorphism. It then follows

in particular that for any quasi-isomorphism $F \rightarrow K$ between complexes of strict modules in $\text{Mod}(\mathcal{R})$ with strict cohomology sheaves, the reduction $F/zF \rightarrow K/zK$ is still a quasi-isomorphism. Taking $F \rightarrow K$ to be a flat resolution of our given complex, we get the claimed isomorphism in the derived category. \square

6.c. The Fourier-Mukai transform for twistor modules. Now let $X = A$ be an abelian variety. We want to interpret the Fourier-Mukai transform of section 3.b as the restriction to $z = 1$ of a transform for twistor modules. In doing so we need to replace the moduli space A^\natural of flat rank one connections by a twistor deformation $E(A)$, the moduli space of generalized flat rank one connections. Here by a *generalized connection* or more specifically by a λ -*connection* on a coherent vector bundle $\mathcal{E} \in \text{Coh}(\mathcal{O}_A)$ we mean a \mathbb{C} -linear morphism $\nabla : \mathcal{E} \rightarrow \Omega_A^1 \otimes_{\mathcal{O}_A} \mathcal{E}$ satisfying

$$\nabla(f \cdot s) = df \otimes \lambda s + f \cdot \nabla s$$

for all $s \in \mathcal{E}(U)$, $f \in \mathcal{O}_A(U)$ on any open subset $U \subseteq A$. We say that ∇ is *flat* if its square vanishes. This notion of a generalized flat connection makes sense for any holomorphic function $\lambda \in H^0(A, \mathcal{O}_A)$. For the constant function $\lambda = 1$ we recover the usual notion of a flat connection while for $\lambda = 0$ we get a Higgs field.

The target space of our Fourier-Mukai transform for twistor modules will be the moduli space $E(A)$ of pairs consisting of a line bundle $\mathcal{L} \in \hat{A} = \text{Pic}^\circ(A)$ and a generalized flat connection $\nabla : \mathcal{L} \rightarrow \Omega_A^1 \otimes_{\mathcal{O}_A} \mathcal{L}$. For more on this moduli space we refer to [43, sect. 10]. Recall that it is a torsor over the dual abelian variety \hat{A} via the morphism

$$E(A) \longrightarrow \hat{A} = \text{Pic}^\circ(A), \quad (\mathcal{L}, \nabla) \mapsto \mathcal{L}$$

that forgets the connection, and that the value of the parameter λ of the generalized flat connection is given by a morphism $\lambda : E(A) \rightarrow \mathbb{A}^1$ so that the moduli spaces of flat connections in the usual sense and of Higgs fields on line bundles arise as the fibres

$$\lambda^{-1}(1) = A^\natural \quad \text{and} \quad \lambda^{-1}(0) = \hat{A} \times V \quad \text{for} \quad V = H^0(A, \Omega_A^1).$$

By [43, lemma 10.5] the pullback of the Poincaré line bundle to $A \times E(A)$ has a universal relative generalized flat connection

$$\nabla : \mathcal{P}|_{E(A)} \longrightarrow \Omega_{A \times E(A)/E(A)}^1 \otimes_{\mathcal{O}_{A \times E(A)}} \mathcal{P}|_{E(A)}.$$

Taking the relative de Rham complex for the projection $p : A \times E(A) \rightarrow E(A)$, we define the Fourier-Mukai transform on the derived category of Rees modules as in loc. cit. by

$$\begin{aligned} \text{FM} : D^b(\mathcal{R}_A^{an}) &\longrightarrow D^b(\mathcal{O}_{E(A)}^{an}), \\ \mathcal{M} &\mapsto Rp_* \text{DR}_{A \times E(A)/E(A)}((id \times \lambda)^* \mathcal{M} \otimes_{\mathcal{O}_{A \times E(A)}^{an}} (\mathcal{P}|_{E(A)}, \nabla)). \end{aligned}$$

Notice however that we work analytically because we are ultimately interested in twistor modules. For $\mathcal{M} \in \text{Hol}(\mathcal{R}_A^{an})$ the existence of a good filtration over the preimage of each sufficiently small open subset of the affine line implies that the complex $\text{FM}(\mathcal{M})$ has coherent cohomology sheaves. In particular, we obtain a functor

$$\text{FM} : D_A \longrightarrow D_{coh}^b(\mathcal{O}_{E(A)}^{an})$$

on the category D_A of pure twistor complexes of weight zero from section 6.b.

The generic vanishing property (1.1) for the Fourier-Mukai transform on $\text{Hol}(\mathcal{D}_A)$ carries over to the case of twistor modules as follows. Let us say that a subvariety of $E(A)$ is a *linear subvariety* if it has the form $E(B) \hookrightarrow E(A)$ where $A \twoheadrightarrow B$ is an epimorphism of abelian varieties. Then for every twistor module $\mathcal{M} \in \text{MT}(A, 0)$ there is an open subset $U \subseteq E(A)$ whose complement is a finite union of translates of proper linear subvarieties such that

$$(6.3) \quad \mathcal{H}^i(\text{FM}(\mathcal{M}))|_U \text{ is } \begin{cases} \text{locally free} & \text{for } i = 0, \\ \text{zero} & \text{for } i \neq 0. \end{cases}$$

The idea of the proof is to transfer the codimension estimates in [43] from A^\natural to all of $E(A)$ by showing that the restriction of the Fourier-Mukai transform to each twistor line has locally free cohomology sheaves; details will be given elsewhere.

6.d. Another fibre functor. By the above twistor reformulation of the vanishing theorem we may perform the constructions from section 3.b not only at $z = 1$ but also at $z = 0$. In this abelianized version our Fourier-Mukai transform reduces to the usual relative Fourier-Mukai transform for coherent sheaves on the abelian scheme $A \times V \rightarrow V$ (see the proof of theorem 6.4).

To see what we have gained from this, let $U \subseteq E(A)$ be the complement of a finite union of translates of proper linear subvarieties, and consider $U_z = U \cap \lambda^{-1}(z)$ for $z \in \mathbb{A}^1(\mathbb{C})$. Let

$$\text{Hol}(A, U_1)^{ss} \subset \text{Hol}(\mathcal{D}_A, U_1)$$

be the full subcategory of all semisimple holonomic modules $\mathcal{N} \simeq \Xi_{\text{DR}}(\mathcal{M})$ for which the corresponding twistor module $\mathcal{M} \in \text{MT}(A, 0)$ satisfies the vanishing condition (6.3) and for which the Gauss map

$$\gamma : \text{Supp}(\Xi_{\text{Dol}}(\mathcal{M})|_{A \times U_0}) \subset A \times U_0 \xrightarrow{p} U_0$$

is a finite morphism. We view $\text{Hol}(\mathcal{D}_A, U_1)^{ss}$ as a tensor category with respect to the tensor product obtained from the convolution product by discarding all negligible direct summands. Then the above discussion leads to the following result, where for objects $\mathcal{E} \in \text{VB}(A, U_0)$ in the tensor category of section 4.b and for $u \in U_0(\mathbb{C})$ we denote by

$$X_u(\mathcal{E}) = \langle a \in A(\mathbb{C}) \mid (a, u) \in \text{Supp}(\mathcal{E}) \rangle \subset A(\mathbb{C})$$

the group generated by the points in the fibre of the corresponding Gauss map.

Theorem 6.4. *For $U \subseteq E(A)$ as above, we have a commutative diagram of tensor functors*

$$\begin{array}{ccc} \text{Hol}(\mathcal{D}_A, U_1)^{ss} & \xrightarrow{\mathcal{H}^0(\text{FM}(-))|_{U_0}} & \text{Coh}(\mathcal{O}_{U_0}^{an}) \\ & \searrow \exists \Phi & \nearrow p_* \\ & \text{VB}(A, U_0) & \end{array}$$

Taking the fibre at $u \in U_0(\mathbb{C})$, we thus obtain for any $\mathcal{M} \in \text{Hol}(\mathcal{D}_A, U_1)^{ss}$ an embedding

$$\text{Hom}(X_u, \mathbb{G}_m) \hookrightarrow G(\mathcal{M}) \quad \text{where} \quad X_u = X_u(\Phi(\mathcal{M})).$$

Proof. Via the functor Ξ_{DR} we consider $\text{Hol}(\mathcal{D}_A, U_1)^{ss}$ as a full subcategory of $\text{MT}(A, 0)$. Taking the inverse image under the projection $q : A \times U_0 \rightarrow A \times V$ followed by the involution $\iota = id_A \times (-id_V)$ and tensoring with the Poincaré line bundle, we define

$$\Phi = q^* \iota^* (\Xi_{\text{Dol}}(-)) \otimes \mathcal{P}_{|U_0} : \text{Hol}(\mathcal{D}_A, U_1)^{ss} \longrightarrow \text{VB}(A, U_0).$$

The Poincaré line bundle satisfies $\rho^*(\mathcal{P}_{|U_0} \boxtimes \mathcal{P}_{|U_0}) \simeq \varpi^*(\mathcal{P}_{|U_0})$, so proposition 6.1 gives isomorphisms

$$\Phi(\mathcal{M}_1 * \mathcal{M}_2) \simeq \Phi(\mathcal{M}_1) * \Phi(\mathcal{M}_2) \quad \text{for } \mathcal{M}_1, \mathcal{M}_2 \in \text{Hol}(\mathcal{D}_A, U_1)^{ss}$$

and one may check that these satisfy the usual compatibilities with the associativity, commutativity and unit constraints. Thus Φ becomes a tensor functor. We also note that

$$Rp_*(q^*(-) \otimes \mathcal{P}_{|U_0}) : D_{coh}^b(\mathcal{O}_{A \times V}) \longrightarrow D_{coh}^b(\mathcal{O}_{\hat{A} \times V})$$

is the Fourier-Mukai transform between the derived categories of coherent sheaves on the abelian scheme $A \times V \rightarrow V$ and its dual. The twist by the involution ι makes the diagram of tensor functors commutative by the same base change arguments as in [43, prop. 12.1], and the embedding $\text{Hom}(X_u, \mathbb{G}_m) \hookrightarrow G(\mathcal{M})$ then comes from the first part of theorem 4.3. \square

Example 6.5. For semisimple modules $\mathcal{M} \in \text{Hol}(\mathcal{D}_A, U_1)^{ss}$ that do not underlie a pure Hodge module, usually

$$\text{Supp}(\Phi(\mathcal{M})) \neq \text{Char}(\mathcal{M})$$

and the multiplicative subgroup in theorem 6.4 can be strictly bigger than the one in theorem 1.5. Suppose for example $\dim(A) = 1$, and let $f : A^* = A \setminus \{0\} \rightarrow \mathbb{C}$ be a holomorphic function with a pole at the origin. Consider the trivial smooth bundle $\mathcal{C}_{A^*}^\infty$ equipped with the flat connection $\nabla = d + \partial f + \overline{\partial} \overline{f}$ and the constant Hermitian metric $h(1, 1) = 1$. This is a harmonic bundle of rank one in the sense of [32, sect. 1.2], and the corresponding Higgs bundle is the trivial line bundle $\mathcal{O}_{A^*}^{an}$ equipped with the Higgs field $\theta = \partial f$. By loc. cit. our chosen harmonic bundle is the twistor deformation of a unique simple module $\mathcal{M}^* \in \text{Hol}(\mathcal{D}_{A^*})$. The minimal extension $\mathcal{M} \in \text{Hol}(\mathcal{D}_A)$ of the latter has its characteristic variety contained in the zero section plus the fibre over the origin, whereas $\text{Supp}(\Phi(\mathcal{M}))$ contains the graph of the differential $\partial f : A^* \rightarrow V = H^0(A, \Omega_A^1)$. So the multiplicative subgroup from theorem 1.5 is trivial in this case but the one from theorem 6.4 is not.

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